

LOCAL TO GLOBAL ALGORITHMS FOR THE GORENSTEIN ADJOINT IDEAL OF A CURVE

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ABSTRACT. We present new algorithms for computing adjoint ideals of curves and thus, in the planar case, adjoint curves. With regard to terminology, we follow Gorenstein who states the adjoint condition in terms of conductors.

Our main algorithm yields the Gorenstein adjoint ideal \mathfrak{G} of a given curve as the intersection of what we call local Gorenstein adjoint ideals. Since the respective local computations do not depend on each other, our approach is inherently parallel.

Over the rationals, further parallelization is achieved by a modular version of the algorithm which first computes a number of the characteristic p counterparts of \mathfrak{G} and then lifts these to characteristic zero. As a key ingredient, we establish an efficient criterion to verify the correctness of the lift.

Well-known applications are the computation of Riemann-Roch spaces, the construction of points in moduli spaces, and the parametrization of rational curves.

We have implemented different variants of our algorithms together with Mnuk's approach [Mnuk 1997] in the computer algebra system SINGULAR and give timings to compare the performance of the algorithms.

1. INTRODUCTION

In classical algebraic geometry, starting from Riemann's paper on abelian functions [Riemann 1857], the adjoint curves of an irreducible plane curve Γ have been used as an essential tool in the study of the geometry of Γ . The defining property of an *adjoint curve* is that it passes with “sufficiently high” multiplicity through the singularities of Γ . There are several ways of making this precise, developed in classical papers by [Brill and Noether 1874], [Castelnuovo 1890, 1893], and [Petri 1924], and in more recent work by [Gröbner 1941, Gorenstein 1952] and [van der Waerden 1939, Keller 1974]. We refer to [Keller 1965], [Greco and Valabrega 1979], [Greco and Valabrega 1982], and [Ciliberto and Orecchia 1984] for results comparing the different notions: whereas the adjoint condition given by Brill and Noether is more restrictive, the notions of adjoint curves given by the other authors above coincide.

In this paper, we always consider adjoint curves in the less restrictive sense. In fact, we rely on Gorenstein's algebraic definition which states the adjoint condition at a singular point $P \in \Gamma$ by considering the conductor of the local ring $\mathcal{O}_{\Gamma, P}$ in its normalization. It is a well-known consequence of Max Noether's Fundamentalsatz that the adjoint curves of any given degree m cut out, residual to a fixed divisor supported on the singular locus of Γ , a complete linear series. Of fundamental importance is the case $m = \deg \Gamma - 3$ which, as shown by Gorenstein, yields the canonical series.

The ideal generated by the defining forms of the adjoint curves of Γ is called the *adjoint ideal* of Γ . In [Arbarello and Ciliberto 1983], the concept of adjoint ideals is extended to the non-planar case: consider a non-degenerate irreducible curve $\Gamma \subset \mathbb{P}_k^r = \text{Proj}(S)$, and let I be a saturated homogeneous ideal of S which is supported on the singular locus of Γ . Then, roughly speaking, I is an adjoint

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ideal of Γ if its homogeneous elements of degree $m \gg 0$ cut out, residual to a fixed divisor supported on the singular locus, a complete linear series. As pointed out in [Arbarello and Ciliberto 1983], the existence of adjoint ideals is implicit in classical papers: examples are the Castelnuovo adjoint ideal and the Petri adjoint ideal. In [Ciliberto and Orecchia 1984], it is shown that Gorenstein’s condition leads to the largest possible adjoint ideal, containing all other adjoint ideals, and now referred to as the *Gorenstein adjoint ideal* $\mathfrak{G} = \mathfrak{G}(\Gamma)$. See [Ciliberto and Orecchia 1984] for some remarks on how the different concepts of adjoint ideals compare in the non-planar case.

With regard to practical applications, adjoint curves enter center stage in the classical Brill-Noether algorithm for computing Riemann-Roch spaces, which in turn can be used to construct Goppa codes (see [Le Brigand and Risler 1988]). Furthermore, linear series cut out by adjoint curves allow us to construct explicit examples of smooth curves via singular plane models; a typical application is the experimental study of moduli spaces of curves. If the geometric genus of a plane curve Γ is zero, then the adjoint curves of degree $\deg \Gamma - 2$ specify a birational map to a rational normal curve. Based on this, we can find an explicit parametrization of Γ over its field of definition, starting either from the projective line or a conic. See [Böhm 1999, Böhm et al. 2015c] and the implementation in the SINGULAR library [Böhm et al. 2012c]. Algorithms for parametrization, in turn, have applications in computer aided design, for example, to compute intersections of curves with other algebraic varieties. See also [Sendra et al. 2008].

A well-known algorithm for computing the Gorenstein adjoint ideal $\mathfrak{G} = \mathfrak{G}(\Gamma)$ in the planar case is due to [Mnuk 1997]. This algorithm makes use of linear algebra to obtain \mathfrak{G} from an integral basis for the normalization $\bar{k}[C]$, where C is an affine part of Γ containing all singularities of Γ . Efficient ways of finding integral bases rely on Puiseux series techniques (see [van Hoeij 1994], [Böhm et al. 2015a]). This somewhat limits Mnuk’s approach to characteristic zero. The same applies to the algorithm of [El Kahoui and Moussa 2014], which also computes the Gorenstein adjoint ideal of a plane curve from an integral basis of $\bar{k}[C]$. The approach of [Orecchia and Ramella, 2014], on the other hand, is limited to ordinary multiple points.

In this paper, we present a new algorithm for computing \mathfrak{G} . This algorithm is highly efficient and not restricted to the planar case, special types of singularities or to characteristic zero. The basic idea is to compute \mathfrak{G} as the intersection of “local Gorenstein ideals”, one for each singular point of Γ . Each local ideal is obtained via Gröbner bases, starting from a “local contribution” to the normalization $\bar{k}[C]$ at the respective singular point. To find these contributions, we use the algorithm from [Böhm et al. 2012a] which is a local variant of the normalization algorithm designed in [Greuel et al. 2010a]. In practical terms, given any field of definition $L \subset k$, we treat the points in a complete set of conjugate singularities simultaneously.

Our approach is already faster per se. In addition, it can take advantage of handling special classes of singularities in an ad hoc way. Above all, it is inherently parallel. For input over the rationals, further parallelization is achieved by a modular version of our algorithm which first computes a number of characteristic p counterparts of \mathfrak{G} and then lifts these to characteristic zero. To apply the general rational reconstruction scheme from [Böhm et al. 2012d], we prove an efficient criterion to verify the correctness of the lift.

Our paper is organized as follows: In Section 2, we discuss algorithmic normalization. In Section 3, we review the definition of adjoint ideals and some related facts. In Section 4, we describe global algorithmic approaches to obtain \mathfrak{G} . We first discuss Mnuk’s approach. Then we describe a global approach which relies on normalization and Gröbner bases. In Sections 5 and 6, we present our local to global algorithm for finding \mathfrak{G} via normalization and Gröbner bases. Section 7 pays particular attention

to the planar case, commenting on the direct treatment of special types of singularities. In Section 8, we discuss the modular version of our algorithm. Finally, in Section 9, we compare the performance of the different approaches, relying on our implementations in the computer algebra system SINGULAR, and running various examples coming from algebraic geometry.

2. ALGORITHMS FOR NORMALIZATION

We begin with some general remarks on normalization and the role played by the conductor. For these, let A be any reduced Noetherian ring, and let $Q(A)$ be its total ring of fractions. Then $Q(A)$ is again a reduced Noetherian ring. We write

$$\text{Spec}(A) = \{P \subset A \mid P \text{ prime ideal}\}$$

for the *spectrum* of A . The *vanishing locus* of an ideal J of A is the set $V(J) = \{P \in \text{Spec}(A) \mid P \supset J\}$.

The *normalization* of A , written \bar{A} , is the integral closure of A in $Q(A)$. We call A *normalization-finite* if \bar{A} is a finite A -module, and we call A *normal* if $A = \bar{A}$.

We denote by

$$N(A) = \{P \in \text{Spec}(A) \mid A_P \text{ is not normal}\}$$

the *non-normal locus* of A , and by

$$\text{Sing}(A) = \{P \in \text{Spec}(A) \mid A_P \text{ is not regular}\}$$

the *singular locus* of A .

Remark 2.1. Note that $N(A) \subset \text{Sing}(A)$. Equality holds if A is of pure dimension one. Indeed, a Noetherian local ring of dimension one is normal iff it is regular (see [de Jong and Pfister 2000, Thm. 4.4.9]).

Definition 2.2. If $R \subset S$ is an extension of rings, the *conductor* of A in B is

$$\mathcal{C}_{S/R} = \{r \in R \mid rS \subset S\}.$$

Note that $\mathcal{C}_{S/R}$ is the largest ideal of R which is also an ideal of S .

Notation 2.3. If A is a reduced Noetherian ring as above, we write

$$\mathcal{C}_A = \mathcal{C}_{\bar{A}/A} = \{a \in A \mid a\bar{A} \subset A\}.$$

Lemma 2.4. *We have $N(A) \subset V(\mathcal{C}_A)$. Furthermore, A is normalization-finite iff \mathcal{C}_A contains a nonzerodivisor of A . In this case, $N(A) = V(\mathcal{C}_A)$.*

Proof. See [Greuel and Pfister 2008, Lemmas 3.6.1, 3.6.3]. □

Remark 2.5 (Splitting of Normalization). Finding the normalization can be reduced to the case of integral domains: If P_1, \dots, P_s are the minimal primes of A , then

$$\bar{A} \cong \overline{A/P_1} \times \dots \times \overline{A/P_s}$$

(see [de Jong and Pfister 2000, Thm. 1.5.20]).

Remark 2.6. Let k be a field. An *affine k -domain* is a finitely generated k -algebra which is an integral domain. By Emmy Noether's finiteness theorem (see [Eisenbud 1995, Cor. 13.13]), any such domain is normalization-finite, and its normalization is an affine k -domain as well. Geometrically, by gluing, this implies that any integral variety X over k admits a (unique) *normalization map* $\bar{X} \rightarrow X$, where \bar{X} is again an integral variety over k (see, for example, [Liu 2002, Sec. 4.1.2]). Specifically, by Remark 2.1, if Γ is a curve over k , we get the *nonsingular model* $\pi : \bar{\Gamma} \rightarrow \Gamma$.

Now, we briefly discuss algorithmic normalization. We begin by recalling the normalization algorithm of Greuel, Laplagne, and Seelisch [Greuel et al. 2010a], which is an improvement of de Jong's algorithm (see [de Jong 1998], [Decker et al. 1999]). This algorithm, to which we refer as the GLS Algorithm, is based on the normality criterion of Grauert and Remmert. To state this criterion, we need:

Lemma 2.7. *Let A be a reduced Noetherian ring, and let $J \subset A$ be an ideal which contains a nonzerodivisor g of A . Then:*

- (1) *If $\varphi \in \text{Hom}_A(J, J)$, the fraction $\varphi(g)/g \in \overline{A}$ is independent of the choice of g , and φ is multiplication by $\varphi(g)/g$.*
- (2) *There are natural inclusions of rings*

$$A \subset \text{Hom}_A(J, J) \cong \frac{1}{g}(gJ :_A J) \subset \overline{A} \subset Q(A), \quad a \mapsto \varphi_a, \quad \varphi \mapsto \frac{\varphi(g)}{g},$$

where φ_a is multiplication by a .

Proof. See [Greuel and Pfister 2008, Lemmas 3.6.1, 3.6.3]. \square

Proposition 2.8 (Grauert and Remmert Criterion). *Let A be a reduced Noetherian ring, and let $J \subset A$ be a radical ideal which contains a nonzerodivisor g of A and satisfies $V(\mathcal{C}_A) \subset V(J)$. Then A is normal iff $A \cong \text{Hom}_A(J, J)$ via the map which sends a to multiplication by a .*

Proof. See [Grauert and Remmert 1971], [Greuel and Pfister 2008, Prop. 3.6.5]. \square

Definition 2.9. A pair (J, g) as in the proposition is called a *test pair* for A , and J is called a *test ideal* for A .

If k is a field and A is an affine k -domain, then test pairs exist by Lemma 2.4 and Emmy Noether's finiteness theorem. If, in addition, k is perfect, a test pair can be found by applying the Jacobian criterion (see [Eisenbud 1995, Thm. 16.19] for this criterion). In fact, in this case, we may choose the radical of the Jacobian ideal M together with any nonzero element g of M as a test pair. Given a test pair (J, g) , the basic idea of finding \overline{A} is to enlarge A by a sequence of finite extensions of affine k -domains

$$A_{i+1} \cong \text{Hom}_{A_i}(J_i, J_i) \cong \frac{1}{g}(gJ_i :_{A_i} J_i) \subset \overline{A} \subset Q(A),$$

with $A_0 = A$ and $J_i = \sqrt{JA_i}$, until the Grauert and Remmert criterion allows one to stop. According to [Greuel et al. 2010a], each A_i can be represented as a quotient $\frac{1}{d_i}U_i \subset Q(A)$, where $U_i \subset A$ is an ideal and $d_i \in U_i$ is nonzero. In this way, all computations except those of the radicals J_i may be carried through in A .

Example 2.10. For

$$A = \mathbb{C}[x, y] = \mathbb{C}[X, Y]/\langle X^5 - Y^2(Y - 1)^3 \rangle,$$

the radical of the Jacobian ideal is

$$J := \langle x, y(y - 1) \rangle_A,$$

so that we can take (J, x) as a test pair. Then, in its first step, the normalization algorithm yields

$$A_1 = \frac{1}{x}U_1 = \frac{1}{x} \langle x, y(y - 1)^2 \rangle_A.$$

In the next steps, we get

$$A_2 = \frac{1}{x^2}U_2 = \frac{1}{x^2} \langle x^2, xy(y - 1), y(y - 1)^2 \rangle_A$$

and

$$A_3 = \frac{1}{x^3}U_3 = \frac{1}{x^3} \langle x^3, x^2y(y - 1), xy(y - 1)^2, y^2(y - 1)^2 \rangle_A.$$

In the final step, we find that A_3 is normal and, hence, equal to \overline{A} .

Next, we describe a local to global variant of the GLS algorithm, given in [Böhm et al. 2012a], which is a considerable enhancement of the algorithm, and which serves as a motivation for our local to global approach to compute the Gorenstein adjoint ideal. This variant is based on the following two observations from [Böhm et al. 2012a]: First, the normalization \overline{A} can be computed as the sum of local contributions $A \subset A^{(i)} \subset \overline{A}$, and second, local contributions can be obtained efficiently by a local variant of the GLS algorithm. For our purposes here, it is enough to present the relevant results in a special case. Here, as usual, if P is a prime of a ring R , and M is an R -module, we write M_P for the localization of M at $R \setminus P$.

Proposition 2.11. *Let A be an affine k -domain of dimension one, and let $\text{Sing}(A) = \{P_1, \dots, P_s\}$ be its singular locus. For $i = 1, \dots, s$, let an intermediate ring $A \subset A^{(i)} \subset \overline{A}$ be given such that $A_{P_i}^{(i)} = \overline{A}_{P_i}$. Then*

$$\sum_{i=1}^s A^{(i)} = \overline{A}.$$

Proof. See [Böhm et al. 2012a, Prop. 15]. \square

Definition 2.12. A ring $A^{(i)}$ as above is called a *local contribution* to \overline{A} at P_i . It is called a *minimal local contribution* if $A_{P_j}^{(i)} = A_{P_j}$ for $j \neq i$.

The computation of local contributions is based on the modified version of the Grauert and Remmert criterion below:

Proposition 2.13. *Let A be an affine k -domain of dimension one, let $A \subset A'$ be a finite ring extension, let $P \in \text{Sing}(A)$, and let $J' = \sqrt{PA'}$. If*

$$A' \cong \text{Hom}_{A'}(J', J')$$

via the map which sends a' to multiplication by a' , then A'_P is normal.

Proof. See [Böhm et al. 2012a, Prop. 16]. \square

Considering an affine domain A of dimension one over a perfect field k , let $P \in \text{Sing}(A)$. Choose P together with a nonzero element g in P instead of a test pair as in Definition 2.9. Then, proceeding as before, we get a chain of affine k -domains

$$A \subset A_1 \subset \dots \subset A_m \subset \overline{A}$$

such that A_m is a local contribution to \overline{A} at P .

Remark 2.14. Given A as above, a finite ring extension $A \subset A'$, and a prime $P \in \text{Sing}(A)$, let $Q \in \text{Sing}(A)$ be a prime different from P , and let $J' = \sqrt{PA'}$. Then

$$\begin{aligned} \text{Hom}_{A'}(J', J')_Q &\cong \text{Hom}_{A'_Q}(J'_Q, J'_Q) \\ &\cong \text{Hom}_{A'_Q}(A'_Q, A'_Q) \cong A'_Q \end{aligned}$$

(see [Eisenbud 1995, Proposition 2.10]). Inductively, this shows that the algorithm outlined above computes a minimal local contribution to \overline{A} at P . Note that such a contribution is uniquely determined since, by definition, its localization at each $P \in \text{Spec}(A)$ is determined.

Example 2.15. In the case of Example 2.10, there are two singularities $P_1 = \langle x, y \rangle$ and $P_2 = \langle x, y - 1 \rangle$. For P_1 , the local normalization algorithm yields $\overline{A}_{P_1} = (\frac{1}{d_1}U_1)_{P_1}$, where

$$d_1 = x^2 \quad \text{and} \quad U_1 = \langle x^2, y(y-1)^3 \rangle_A.$$

For P_2 , we get $\overline{A}_{P_2} = (\frac{1}{d_2}U_2)_{P_2}$, where

$$d_2 = x^3 \quad \text{and} \quad U_2 = \langle x^3, x^2y^2(y-1), y^2(y-1)^2 \rangle_A.$$

Combining the local contributions, we get

$$\frac{1}{d}U = \frac{1}{d_1}U_1 + \frac{1}{d_2}U_2$$

with $d = x^3$ and

$$U = \left\langle x^3, xy(y-1)^3, x^2y^2(y-1), y^2(y-1)^2 \right\rangle_A.$$

Note that U coincides with the ideal U_3 computed in Example 2.10.

Notation 2.16. In our applications, A will always be the coordinate ring $k[C] = k[X_1, \dots, X_r]/I(C)$ of an integral affine curve $C \subset \mathbb{A}_k^r$ over a perfect field k . Given a point¹ $P \in C$, by abuse of notation, if $I \subset k[x_1, \dots, x_r]$ is an ideal properly containing $I(C)$, we will write I_P for the ideal of the local ring $\mathcal{O}_{C,P}$ obtained by mapping I to $k[C]$ and localizing at P . Likewise for the homogeneous localization of a homogeneous ideal in the projective case.

3. ADJOINT IDEALS

Let k be a field, and let $\Gamma \subset \mathbb{P}_k^r$ be an integral non-degenerate projective curve. Write $S = k[X_0, \dots, X_r]$ for the homogeneous coordinate ring of \mathbb{P}_k^r , $I(\Gamma) \subset S$ for the homogeneous ideal of Γ , $k[\Gamma] = S/I(\Gamma)$ for the homogeneous coordinate ring of Γ , and $\text{Sing}(\Gamma)$ for the singular locus of Γ .

Let $\pi : \bar{\Gamma} \rightarrow \Gamma$ be the normalization map, let P be a point of Γ , and let $\mathcal{O}_{\Gamma,P}$ be the local ring of Γ at P . Then the normalization $\bar{\mathcal{O}}_{\Gamma,P}$ is a semi-local ring whose maximal ideals correspond to the points of $\bar{\Gamma}$ lying over P . Furthermore, $\bar{\mathcal{O}}_{\Gamma,P}$ is finite over $\mathcal{O}_{\Gamma,P}$ and, thus, a finite-dimensional k -vector space. The dimension

$$\delta_P(\Gamma) = \delta(\mathcal{O}_{\Gamma,P}) = \dim_k \bar{\mathcal{O}}_{\Gamma,P} / \mathcal{O}_{\Gamma,P}$$

is called the *delta invariant* of Γ at P . The *arithmetic genus* of Γ is $p_a(\Gamma) = 1 - P_\Gamma(0)$, where P_Γ is the Hilbert polynomial of $k[\Gamma]$. Making use of the (global) *delta invariant*

$$\delta(\Gamma) = \sum_{P \in \text{Sing}(\Gamma)} \delta_P(\Gamma)$$

of Γ , the *geometric genus* $p(\Gamma)$ of Γ is given by

$$p(\Gamma) = p(\bar{\Gamma}) = p_a(\Gamma) - \delta(\Gamma)$$

(see [Hironaka 1957]). If Γ is a plane curve of degree n , we have $p_a(\Gamma) = \binom{n-1}{2}$.

Following the presentation in [Chiarli 1984], we now recall the definition and characterization of adjoint ideals due to [Arbarello and Ciliberto 1983] and [Ciliberto and Orecchia 1984]. Let $I = \bigoplus_{m \geq 0} I_m \subset S = k[X_0, \dots, X_r]$ be a saturated homogeneous ideal properly containing $I(\Gamma)$. Pulling back $\text{Proj}(S/I)$ via π , we get an effective divisor $\Delta(I)$ on $\bar{\Gamma}$. Let H be a divisor on $\bar{\Gamma}$ given as the pullback of a hyperplane in \mathbb{P}_k^r . Then, since any divisor on $\bar{\Gamma}$ cut out by a homogeneous polynomial in I is of the form $D + \Delta(I)$ for some effective divisor D , we have natural linear maps

$$\varrho_m : I_m \rightarrow H^0(\bar{\Gamma}, \mathcal{O}_{\bar{\Gamma}}(mH - \Delta(I))),$$

for all $m \geq 0$.

Remark 3.1. Consider the exact sequence

$$0 \rightarrow \tilde{I}\mathcal{O}_\Gamma \rightarrow \pi_*(\tilde{I}\mathcal{O}_{\bar{\Gamma}}) \rightarrow \mathcal{F} \rightarrow 0,$$

where \tilde{I} is the ideal sheaf associated to I , and \mathcal{F} is the cokernel. Taking global sections, we get, for $m \gg 0$, the exact sequence

$$0 \rightarrow H^0(\Gamma, \tilde{I}\mathcal{O}_\Gamma(m)) \rightarrow H^0(\bar{\Gamma}, \tilde{I}\mathcal{O}_{\bar{\Gamma}}(mH)) \rightarrow H^0(\Gamma, \mathcal{F}) \rightarrow 0.$$

¹The term *point* will always refer to a closed point.

Indeed, \mathcal{F} has finite support and, since the normalization map π is finite, we have $H^0(\bar{\Gamma}, \tilde{I}\mathcal{O}_{\bar{\Gamma}}(mH)) \cong H^0(\Gamma, \pi_*(\tilde{I}\mathcal{O}_{\bar{\Gamma}})(m))$. Since $\tilde{I}\mathcal{O}_{\bar{\Gamma}}(mH) = \mathcal{O}_{\bar{\Gamma}}(mH - \Delta(I))$ and, for $m \gg 0$, $H^0(\Gamma, \tilde{I}\mathcal{O}_{\Gamma}(m)) = I_m/I(\Gamma)_m$, we get, for $m \gg 0$, the exact sequence

$$0 \rightarrow I_m/I(\Gamma)_m \xrightarrow{\varrho_m} H^0(\bar{\Gamma}, \mathcal{O}_{\bar{\Gamma}}(mH - \Delta(I))) \rightarrow H^0(\Gamma, \mathcal{F}) \rightarrow 0.$$

In particular, for $m \gg 0$,

$$\ker(\varrho_m) = I(\Gamma)_m.$$

Definition 3.2. With notation and assumptions as above, the ideal I is called an *adjoint ideal* of Γ if the maps

$$\varrho_m : I_m \rightarrow H^0(\bar{\Gamma}, \mathcal{O}_{\bar{\Gamma}}(mH - \Delta(I)))$$

are surjective for m large enough.

As already remarked in the introduction, the existence of adjoint ideals is classical. Locally, adjoint ideals are characterized by the following criterion:

Theorem 3.3. *The ideal I is an adjoint ideal of Γ iff $I_P = I_P \overline{\mathcal{O}_{\Gamma, P}}$ for all $P \in \text{Sing}(\Gamma)$.*

Proof. Using the notation of Remark 3.1, we have, for $m \gg 0$,

$$\dim_k \text{coker } \varrho_m = h^0(\Gamma, \mathcal{F}) = \sum_{P \in \text{Sing}(\Gamma)} \ell(I_P \overline{\mathcal{O}_{\Gamma, P}}/I_P).$$

Hence, ϱ_m is surjective iff $I_P \overline{\mathcal{O}_{\Gamma, P}} = I_P$ for all $P \in \text{Sing}(\Gamma)$. \square

Corollary 3.4. *If I is an adjoint ideal of Γ and $P \in \text{Sing}(\Gamma)$, then $I_P \subsetneq \mathcal{O}_{\Gamma, P}$.*

Proof. Suppose $I_P = \mathcal{O}_{\Gamma, P}$. Then $I_P \subsetneq I_P \overline{\mathcal{O}_{\Gamma, P}}$, a contradiction to Theorem 3.3. \square

Corollary 3.5. *The support of $\text{Proj}(S/I)$ contains $\text{Sing}(\Gamma)$.*

Proof. Follows immediately from Corollary 3.4. \square

Theorem 3.6. *There is a unique largest homogeneous ideal $\mathfrak{G} \subset S$ which satisfies*

$$\mathfrak{G}_P = \mathcal{C}_{\mathcal{O}_{\Gamma, P}} \text{ for all } P \in \text{Sing}(\Gamma).$$

The ideal \mathfrak{G} is an adjoint ideal of Γ containing all other adjoint ideals of Γ . In particular, \mathfrak{G} is saturated and $\text{Proj}(S/\mathfrak{G})$ is supported on $\text{Sing}(\Gamma)$.

Proof. For the conductor ideal sheaf $\mathcal{C} = \text{Ann}_{\mathcal{O}_{\Gamma}}(\pi_*\mathcal{O}_{\bar{\Gamma}}/\mathcal{O}_{\Gamma})$ on Γ , we have $\mathcal{C}_P = \mathcal{C}_{\mathcal{O}_{\Gamma, P}}$ for all $P \in \Gamma$. If $j : \Gamma \rightarrow \mathbb{P}_k^r$ is the inclusion, then the graded S -module $\mathfrak{G} = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}_k^r, j_*\mathcal{C}(n))$ associated to $j_*\mathcal{C}$ is the unique largest homogeneous ideal with $\mathfrak{G}_P = \mathcal{C}_{\mathcal{O}_{\Gamma, P}}$ for all $P \in \text{Sing}(\Gamma)$. By Theorem 3.3 and the properties of the conductor, \mathfrak{G} is an adjoint ideal. Moreover, if I is any other adjoint ideal, then $I_P \subset \mathfrak{G}_P$ for all $P \in \Gamma$, hence $I \subset \mathfrak{G}$. \square

Definition 3.7. With notation as in Theorem 3.6, the ideal \mathfrak{G} is called the *Gorenstein adjoint ideal* of Γ . We also write $\mathfrak{G}(\Gamma) = \mathfrak{G}$.

For repeated subsequent use, we introduce the following notation:

Notation 3.8. Let $\Gamma \subset \mathbb{P}_k^r$ be a curve as above with Gorenstein adjoint ideal \mathfrak{G} . Let C be the affine part of Γ with respect to the chart

$$\mathbb{A}_k^r \hookrightarrow \mathbb{P}_k^r, (X_1, \dots, X_r) \mapsto (1 : X_1 : \dots : X_r)$$

let $I(C) \subset k[X_1, \dots, X_r]$ be the ideal of C , let $k[C] = k[x_1, \dots, x_r] = k[X_1, \dots, X_r]/I(C)$ be its coordinate ring, and let $\text{Sing}(C)$ be its set of singular points.

Proposition 3.9. *Assume $\Gamma \subset \mathbb{P}_k^r$ is a curve as in Notation 3.8, with affine part C . Let $\overline{\mathfrak{G}}$ be the ideal of $k[C]$ obtained by dehomogenizing \mathfrak{G} with respect to X_0 and mapping the result to $k[C]$. Then*

$$\overline{\mathfrak{G}} = \mathcal{C}_{k[C]}.$$

If Γ has no singularities at infinity² and $\mathcal{C}_{k[C]} = \langle g_i(x_1, \dots, x_r) \mid i \rangle_{k[C]}$ with polynomials $g_i \in k[X_1, \dots, X_r]$, then \mathfrak{G} is the homogenization of

$$\langle g_i(X_1, \dots, X_r) \mid i \rangle_{k[X_1, \dots, X_r]} + I(C)$$

with respect to X_0 .

Proof. The first statement is obtained by localizing at the points of C :

$$\overline{\mathfrak{G}}_P = \mathcal{C}_{\mathcal{O}_{C,P}} = (\mathcal{C}_{k[C]})_P \text{ for each } P \in C.$$

Here, the first equality is clear from the definition of \mathfrak{G} (see Theorem 3.6). The second equality holds since forming the conductor commutes with localization since $k[C]$ is normalization-finite (see [Zariski and Samuel 1975, Ch. V, § 5]).

The second statement of the proposition follows from the first one since there are no singularities at infinity, \mathfrak{G} is saturated, and the support of \mathfrak{G} is contained in C . \square

We take a moment to specialize to plane curves.

Remark 3.10. Assume Γ is a plane curve. Then, by Max Noether's Fundamentalsatz, the maps $\varrho_m : \mathfrak{G}_m \rightarrow H^0(\overline{\Gamma}, \mathcal{O}_{\overline{\Gamma}}(mH - \Delta(\mathfrak{G})))$ are surjective for *all* m . Referring to each homogeneous polynomial in \mathfrak{G} not contained in $I(\Gamma)$ as an *adjoint curve* to Γ , this means that residual to $\Delta(\mathfrak{G})$, the adjoint curves of any degree m cut out the complete linear series $\mathcal{A}_m = |mH - \Delta(\mathfrak{G})|$. See [van der Waerden 1939, § 49].

Theorem 3.11. *Assume Γ is a plane curve of degree n . Then, residual to $\Delta(\mathfrak{G})$, the elements of \mathfrak{G}_{n-3} cut out the complete canonical linear series. Equivalently,*

$$\deg \Delta(\mathfrak{G}) = 2\delta(\Gamma). \quad (3.1)$$

Proof. See [Gorenstein 1952, Thm. 9]. \square

Recall that the dimension of the canonical linear series is $\dim \mathcal{A}_{n-3} = p(\Gamma) - 1$.

Remark 3.12. Assume Γ is a plane curve of degree n . If $p(\Gamma) = 0$, that is, Γ is rational, then $\dim \mathcal{A}_{n-2} = \deg \mathcal{A}_{n-2} = n - 2$. In this case, the image of Γ under \mathcal{A}_{n-2} is a rational normal curve $\Gamma_{n-2} \subset \mathbb{P}_k^{n-2}$ of degree $n - 2$. Via the birational morphism $\Gamma_{n-2} \rightarrow \Gamma$, the problem of parametrizing Γ is reduced to parametrizing the smooth curve Γ_{n-2} . For the latter, we may successively decrease the degree of the rational normal curve by 2 via the anti-canonical linear series. This yields an isomorphism from Γ_{n-2} either to \mathbb{P}^1 or to a plane conic, depending on whether n is odd or even. If Γ is defined by an equation over a subfield $L \subset k$, then all computations considered so far take place over the coefficient field L . Parametrizing the conic, however, may require a quadratic field extension, depending on whether the conic contains an L -rational point or not. See [Böhm 1999] and [Böhm et al. 2015c] for details.

By generalizing the formula in Theorem 3.11, we now derive a characterization of adjoint ideals, which is also valid in the non-planar case. We use the following notation: If $I \subset S$ is a homogeneous ideal, write $\deg I = \deg \text{Proj}(S/I)$. That is, $\deg I$ is $(\dim I - 1)!$ times the leading coefficient of the Hilbert polynomial of S/I .

²If k is infinite, this assumption can always be achieved by a projective automorphism defined over k . Otherwise, we may have to replace k by an extension field of k .

Lemma 3.13. *Let $I \subset S$ be a saturated homogeneous ideal with $I(\Gamma) \subsetneq I$. Then*

$$\deg \Delta(I) \leq \deg I + \delta(\Gamma),$$

and I is an adjoint ideal of Γ iff

$$\deg \Delta(I) = \deg I + \delta(\Gamma).$$

Proof. Let $P_\Gamma(t) = (\deg \Gamma) \cdot t - p_a(\Gamma) + 1$ be the Hilbert polynomial of $k[\Gamma]$. Denote by I_Γ the image of I in $k[\Gamma]$. Then, for $m \gg 0$,

$$\deg I = \dim_k(S_m/I_m) = \dim_k(k[\Gamma]_m/(I_\Gamma)_m) = P_\Gamma(m) - \dim_k(I_\Gamma)_m.$$

Moreover, by Remark 3.1, $h^0(\overline{\Gamma}, \mathcal{O}_{\overline{\Gamma}}(mH - \Delta(I))) = \dim_k(I_\Gamma)_m + h^0(\Gamma, \mathcal{F}) \geq \dim_k(I_\Gamma)_m$ for $m \gg 0$. Hence, by Riemann-Roch, for $m \gg 0$, we have

$$\begin{aligned} (\deg \Gamma) \cdot m - \deg \Delta(I) &= \deg |mH - \Delta(I)| = \dim |mH - \Delta(I)| + p(\Gamma) \\ &\geq \dim_k(I_\Gamma)_m - 1 + p(\Gamma) \\ &= P_\Gamma(m) - \deg I - 1 + p(\Gamma) \\ &= (\deg \Gamma) \cdot m - \delta(\Gamma) - \deg I. \end{aligned}$$

Here, we use that $|mH - \Delta(I)|$ is nonspecial for large m by reason of its degree. Equality holds iff ϱ_m is surjective. \square

Remark 3.14. In the case where Γ is a plane curve and $I = \mathfrak{G}$ is its Gorenstein adjoint ideal, Lemma 3.13 shows that Equation (3.1) may be rewritten as

$$\deg \mathfrak{G} = \delta(\Gamma). \quad (3.2)$$

Note that (3.1) and (3.2) may not hold in the non-planar case:

Example 3.15 ([de Jong and Pfister 2000, Example 5.2.5]). Let $\Gamma \subset \mathbb{P}_{\mathbb{C}}^3$ be the image of the parametrization

$$\mathbb{P}_{\mathbb{C}}^1 \longrightarrow \mathbb{P}_{\mathbb{C}}^3, (s : t) \mapsto (s^5 : t^3 s^2 : t^4 s : t^5).$$

Then Γ has exactly one singularity at $(1 : 0 : 0 : 0)$. Furthermore, $p(\Gamma) = 0$ and $p_a(\Gamma) = 2$, hence $\delta(\Gamma) = 2$. However, $\mathfrak{G} = \langle X_1, X_2, X_3 \rangle \subset \mathbb{C}[X_0, \dots, X_3]$, hence $\deg \mathfrak{G} = 1$.

Remark 3.16. If $\Gamma \subset \mathbb{P}_k^r$ is any curve as in Notation 3.8, with affine part C and no singularities at infinity, then it follows from Proposition 3.9 that

$$\deg \mathfrak{G} = \dim_k(k[C]/\mathcal{C}_{k[C]}) = \sum_{P \in \text{Sing}(C)} \dim_k(\mathcal{O}_{C,P}/\mathcal{C}_{\mathcal{O}_{C,P}}).$$

Lemma 3.17. *If $\text{char } k = 0$, then $\dim_k(\mathcal{O}_{\Gamma,P}/\mathcal{C}_{\mathcal{O}_{\Gamma,P}}) \leq \delta_P(\Gamma)$ for any point $P \in \Gamma$.*

Proof. Since normalization commutes with base change, this follows from the case $k = \mathbb{C}$ proved in [Greuel 1982, 2.4]. \square

Now recall that a point $P \in \text{Sing}(\Gamma)$ is called a *Gorenstein singularity* if

$$\dim_k(\mathcal{O}_{\Gamma,P}/\mathcal{C}_{\mathcal{O}_{\Gamma,P}}) = \delta_P(\Gamma).$$

Example 3.18. Plane curve singularities are Gorenstein (see, for example, [de Jong and Pfister 2000, Corollary 5.2.9]).

Corollary 3.19. *We have:*

- (1) *If $\text{char } k = 0$, then $\deg \mathfrak{G} \leq \delta(\Gamma)$.*
- (2) *If Γ has only Gorenstein singularities, then*

$$\deg \mathfrak{G} = \delta(\Gamma) \quad \text{and} \quad \deg \Delta(\mathfrak{G}) = 2\delta(\Gamma).$$

Proof. This is clear from the discussion above. \square

In the case of arbitrary singularities, we will make use of the equality

$$\deg \mathfrak{G} = \deg \Delta(\mathfrak{G}) - \delta(\Gamma)$$

to compute $\deg \mathfrak{G}$ without actually knowing \mathfrak{G} , and apply this in the final verification step of our modularized adjoint ideal algorithm. To this end, if $k = \mathbb{C}$ and Γ is defined over the rationals, we will present a modular approach to computing $\deg \Delta(\mathfrak{G})$, and we will use standard techniques to compute $\delta(\Gamma)$. In fact, for the latter, first note that the delta invariant of Γ differs from that of a plane model of Γ by the quantity $p_a(\Gamma) - \binom{\deg \Gamma - 1}{2}$. The delta invariant of a plane curve, in turn, can be computed locally at the singular points, either from the semigroups of values of the analytic branches of the singularity (see [de Jong and Pfister 2000], [Greuel et al. 2007]), or from a formula relating the local delta invariant to the Milnor number (see Remark 7.3 in Section 7 below).

Remark 3.20. Note that computing $\deg \Delta(\mathfrak{G})$ also means to compute the dimension $\dim_k (\overline{k[C]}/\mathcal{C}_{k[C]})$: Given $\Gamma \subset \mathbb{P}_k^r$ as in Notation 3.8, with affine part C and no singularities at infinity, we have

$$\begin{aligned} \deg \Delta(\mathfrak{G}) &= \delta(\Gamma) + \deg \mathfrak{G} \\ &= \dim_k (\overline{k[C]}/k[C]) + \dim_k (k[C]/\mathcal{C}_{k[C]}) \\ &= \dim_k (\overline{k[C]}/\mathcal{C}_{k[C]}). \end{aligned}$$

We are now ready to address the computation of the Gorenstein adjoint ideal. Using Proposition 3.9, one way of finding \mathfrak{G} is to apply the global algorithm presented in Section 4.2 below, starting from the normalization $\overline{k[C]}$. The normalization, in turn, can be found by combining the minimal local contributions to $\overline{k[C]}$ at the singular points via Proposition 2.11. As it turns out, however, it is more efficient to directly compute local Gorenstein adjoint ideals at the singular points, and get \mathfrak{G} as their intersection. This will be discussed in Sections 5 and 6.

Remark 3.21. In applications, $\Gamma \subset \mathbb{P}_k^r$ is often defined over a perfect subfield $k' \subset k$ (for example, $k' = \mathbb{Q}$ and $k = \mathbb{C}$). In such a situation, by base change, $\delta(\Gamma) = \delta(\Gamma(k'))$. Moreover, since the algorithms in Sections 5 and 6 rely on Gröbner bases, and Buchberger's algorithm for computing Gröbner bases does not leave the ground field, $\mathfrak{G}(\Gamma) = \mathfrak{G}(\Gamma(k'))K[X_0, \dots, X_n]$, and generators can be found by computations over k' .

4. GLOBAL APPROACHES

4.1. Computing the conductor via the trace matrix. We will require some facts from classical ideal theory (see [Zariski and Samuel 1975, Ch. V] for details and proofs): Let R be an integral domain, and let $K = Q(R)$ be its quotient field. A *fractionary ideal* of R is an R -submodule \mathfrak{b} of K admitting a common denominator: there is an element $0 \neq d \in R$ such that $d\mathfrak{b} \subset R$.

Example 4.1. The extensions A_i computed by the normalization algorithms from Section 2 are fractionary ideals of the given affine domain A .

If $\mathfrak{b}, \mathfrak{b}'$ are two fractionary ideals of R , with \mathfrak{b}' nonzero, then $\mathfrak{b} : \mathfrak{b}' = \{z \in K \mid z\mathfrak{b}' \subset \mathfrak{b}\}$ is a fractionary ideal of R as well. A fractionary ideal \mathfrak{b} of R is *invertible* if there is a fractionary ideal \mathfrak{b}' of R such that $\mathfrak{b} \cdot \mathfrak{b}' = R$. In this case, \mathfrak{b}' is uniquely determined and equal to $R : \mathfrak{b}$.

Suppose in addition that R is normal. Let K' be a finite separable extension of K , and let R' be an integral extension of R such that $K' = Q(R')$. Moreover, let

$$\mathrm{Tr}_{K'/K} : K' \rightarrow K, z \mapsto \sum_{g \in \mathrm{Gal}(K'/K)} g(z),$$

be the *trace map*. Then the *complementary module*

$$\mathfrak{C}_{R'/R} := \{z \in K' \mid \text{Tr}_{K'/K}(zR') \subset R\}$$

of R' with respect to R is a *fractionary ideal* of R' containing R' . Hence, the *different*

$$\begin{aligned} \mathfrak{D}_{R'/R} &= R' : \mathfrak{C}_{R'/R} = \{z \in K' \mid z\mathfrak{C}_{R'/R} \subset R'\} \\ &= \{z \in K' \mid zx \in R' \text{ for all } x \in K' \text{ with } \text{Tr}_{K'/K}(xR') \subset R\} \end{aligned}$$

of R' over R is a nonzero ideal of R' .

Now, keeping our assumptions, we focus on the case where R is a Dedekind domain, and where R' is the integral closure of R in K' . Then R' is a Dedekind domain as well, which implies that every nonzero fractionary ideal of R' is invertible. On the other hand, by the primitive element theorem, there is an element $y \in R'$ with $K' = K(y)$. Denote by $f(Y) \in K[Y]$ the minimal polynomial of y over K . Then, as shown in [Zariski and Samuel 1975, Ch. V],

$$f'(y)R' = \mathcal{C}_{R'/R[y]}\mathfrak{D}_{R'/R},$$

hence

$$\mathcal{C}_{R'/R[y]} = f'(y)\mathfrak{C}_{R'/R}. \quad (4.1)$$

We now fix the following setup:

Notation 4.2. Let k be a perfect field. Let $\Gamma \subset \mathbb{P}_k^2$ be a plane curve of degree n defined by an irreducible polynomial $F \in k[X, Y, Z]$. Suppose that Γ has no singularities at infinity with respect to the affine chart

$$\mathbb{A}_k^2 \hookrightarrow \mathbb{P}_k^2, (X, Y) \mapsto (1 : X : Y),$$

and that the equation $f \in k[X, Y]$ of the affine part C of Γ is monic in Y .

Write $k[C] = k[x, y] = k[X, Y]/\langle f(X, Y) \rangle$ for the affine coordinate ring of C and

$$k(C) = k(x, y) = k(X)[Y]/\langle f(X, Y) \rangle$$

for its function field. Then x is a separating transcendence basis of $k(C)$ over k , and y is integral over $k[x]$, with integral equation $f(x, y) = 0$. In particular, $k[C]$ is integral over $k[x]$, which implies that $\overline{k[C]}$ coincides with the integral closure $\overline{k[x]}$ of $k[x]$ in $k(C)$. Furthermore, $\overline{k[C]}$ is a free $k[x]$ -module of rank

$$n := \deg_y(f) = [k(C) : k(x)].$$

Definition 4.3. An *integral basis* for $\overline{k[C]}$ is a set b_0, \dots, b_{n-1} of free generators for $\overline{k[C]}$ over $k[x]$:

$$\overline{k[C]} = k[x]b_0 \oplus \dots \oplus k[x]b_{n-1}.$$

Remark 4.4. Since $k(C) = k(x, y) = k(X)[Y]/\langle f \rangle$, any element $\alpha \in k(C)$ can be represented as a polynomial in $k(X)[Y]$ of degree less than $n = \deg f$. Hence, one can associate to α a well-defined degree $\deg_y(\alpha)$ in y and a smallest common denominator in $k[x]$ of the coefficients of α . In particular, $\overline{k[C]}$ has an integral basis (b_i) in triangular form, that is, with $\deg_y(b_i) = i$, for $i = 0, \dots, n-1$. If not stated otherwise, all integral bases will be of this form. In principle, such a basis can be found by applying one of the normalization algorithms discussed earlier. However, in the characteristic zero case, methods relying on Puiseux series techniques are much more efficient (see [Böhm et al. 2015a] and [van Hoeij 1994]).

Example 4.5. An integral basis for the curve from Example 2.10 is given below:

$$1, y, \frac{y(y-1)}{x}, \frac{y(y-1)^2}{x^2}, \frac{y^2(y-1)^2}{x^3}.$$

Using Proposition 3.9 and Equation (4.1), with $R = k[x]$, $R' = \overline{k[C]}$, $K = k(x)$, and $K' = k(C)$, we get Algorithm 1.

Algorithm 1 Gorenstein adjoint ideal via linear algebra (see [Mnuk 1997])

Input: A plane curve Γ with affine part C as in Notation 4.2.

Output: The Gorenstein adjoint ideal \mathfrak{G} of Γ .

- 1: Compute an integral basis $(b_i)_{i=0,\dots,n-1}$ for $\overline{k[C]}$.
- 2: Compute the (symmetric and invertible) trace matrix

$$T = (\mathrm{Tr}_{k(C)/k(x)}(b_i b_j))_{i,j=0,\dots,n-1} \in k(x)^{n \times n}.$$

- 3: Compute a decomposition $L \cdot R = P \cdot T$, where L is left triangular matrix with diagonal entries equal to one, R is a right triangular matrix, and P is a permutation matrix.
- 4: For $j = 0, \dots, n-1$, use forward and backward substitution to compute

$$\eta_j = \sum_{i=0}^{n-1} s_{ij} b_i,$$

where $(s_{ij}) = T^{-1}$. The η_j are $k[x]$ -module generators for $\mathfrak{C}_{\overline{k[C]}/k[x]}$. By (4.1),

$$\mathcal{C}_{k[C]} = \langle \frac{\partial f}{\partial Y}(x, y) \eta_j \mid j = 0, \dots, n-1 \rangle.$$

- 5: Let \mathcal{C} be the ideal of $k[X, Y]$ generated by representatives of minimal y -degree of the $\frac{\partial f}{\partial Y}(x, y) \eta_j$, $j = 0, \dots, n-1$.
 - 6: **return** the homogenization of \mathcal{C} with respect to X_0 .
-

Remark 4.6. To compute an integral basis via Puiseux series in the characteristic zero case, we temporarily may have to pass to an algebraic extension field of k .

Example 4.7. The curve $\Gamma \subset \mathbb{P}_{\mathbb{C}}^2$ from Example 2.10 with affine equation

$$X^5 - Y^2(1 - Y)^3 = 0$$

has a singularity of type A_4 at $(0, 0)$ and a 3-fold point of type E_8 at $(0, 1)$. From the integral basis

$$b_0 = 1, \quad b_1 = y, \quad b_2 = \frac{y(y-1)}{x}, \quad b_3 = \frac{y(y-1)^2}{x^2}, \quad \text{and} \quad b_4 = \frac{y^2(y-1)^2}{x^3}$$

given in Example 4.5, we compute the trace matrix

$$T = \begin{pmatrix} 5 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & -5x^2 \\ 0 & 0 & 0 & -5x^2 & -3x \\ 0 & 0 & -5x^2 & -3x & 0 \\ 0 & -5x^2 & -3x & 0 & 0 \end{pmatrix},$$

which yields by forward and backward substitution

$$\mathcal{C}_{\mathbb{C}[C]} = \langle x^3, x^2(y-1), xy(x-1), y(y-1)^2 \rangle_{\mathbb{C}[C]}.$$

Homogenization (and primary decomposition) gives

$$\mathfrak{G} = \langle Y, X^2 \rangle \cap \langle X^3, X(Y-Z), (Y-Z)^2 \rangle.$$

Writing \mathfrak{G} as the intersection of simpler ideals corresponding to the singularities motivates the local to global approach discussed in Sections 5 and 6 below, where \mathfrak{G} will be found as the intersection of local Gorenstein ideals.

4.2. Computing the adjoint ideal via ideal quotients. The algorithm presented in what follows relies on normalization and ideal quotients. It is not limited to plane curves.

Proposition 4.8. *Let $\Gamma \subset \mathbb{P}_k^r$ be a curve with affine part C as in Notation 3.8. Write $\overline{k[C]} = \frac{1}{d}U$, where $U \subset k[C]$ is an ideal and $d \in U$ is nonzero. Then the conductor is*

$$\mathcal{C}_{k[C]} = \langle d \rangle_{k[C]} : U.$$

Proof. By definition,

$$\begin{aligned} \mathcal{C}_{k[C]} &= \left\{ s \in k[C] \mid s \cdot \overline{k[C]} \subset k[C] \right\} \\ &= \{ s \in k[C] \mid s \cdot g \in \langle d \rangle \text{ for all } g \in U \} \\ &= \langle d \rangle_{k[C]} : U. \end{aligned}$$

□

Using once more Proposition 3.9, we get Algorithm 2.

Algorithm 2 Gorenstein adjoint ideal via ideal quotients

Input: A curve $\Gamma \subset \mathbb{P}_k^r$ with affine part C as in Notation 3.8 and no singularities at infinity.

Output: The Gorenstein adjoint ideal \mathfrak{G} of Γ .

- 1: Normalization: Compute polynomials $d, a_0, \dots, a_s \in k[X_1, \dots, X_r]$ such that the fractions $\frac{a_i(x_1, \dots, x_r)}{d(x_1, \dots, x_r)}$ generate $\overline{k[C]}$ as a $k[C]$ -module.
- 2: Compute the ideal quotient

$$\mathcal{C} = (\langle d \rangle + I(C)) : (\langle a_0, \dots, a_s \rangle + I(C)) \subset k[X_1, \dots, X_r].$$

- 3: **return** the homogenization of \mathcal{C} with respect to X_0 .
-

Example 4.9. In Example 2.10,

$$a_0 = X^3, \quad a_1 = X^2Y(Y-1), \quad a_2 = XY(Y-1)^2, \quad a_3 = Y^2(Y-1)^2,$$

and $d = X^3$. Hence,

$$\langle d, f \rangle : \langle a_0, \dots, a_3, f \rangle = \langle X^3, X^2(Y-1), XY(Y-1), Y(Y-1)^2 \rangle.$$

5. A LOCAL TO GLOBAL APPROACH

In this section, motivated by the local to global approach for normalization, we introduce local Gorenstein adjoint ideals of a given curve and show how to find the Gorenstein adjoint ideal \mathfrak{G} as their intersection. Together with the algorithm presented in the next section, where we will show how to compute the local ideals, this yields a local to global approach for finding \mathfrak{G} . As we will see in Section 9, this approach is per se faster than the algorithms discussed so far. In addition, it is well-suited for parallel computations.

We fix the following setup:

Notation 5.1. Let $\Gamma \subset \mathbb{P}_k^r$ be an integral non-degenerate projective curve, and let S be the homogeneous coordinate ring of \mathbb{P}_k^r .

Definition 5.2. Let $W \subset \text{Sing}(\Gamma)$ be a set of singular points of Γ . The *local Gorenstein adjoint ideal* of Γ at W is defined to be the largest homogeneous ideal $\mathfrak{G}(W) \subset S$ which satisfies

$$\mathfrak{G}(W)_P = \mathcal{C}_{\mathcal{O}_{\Gamma, P}} \text{ for all } P \in W. \quad (5.1)$$

For a single point $P \in \text{Sing}(\Gamma)$, we write $\mathfrak{G}(P) := \mathfrak{G}(\{P\})$.

Remark 5.3. Since $\mathfrak{G}(W)$ is the largest homogeneous ideal satisfying (5.1), it is saturated and $\text{Proj}(S/\mathfrak{G}(W))$ is supported on W .

Proposition 5.4. *Let $W \subset \text{Sing}(\Gamma)$. Then*

$$\mathfrak{G}(W) = \bigcap_{P \in W} \mathfrak{G}(P).$$

Proof. This is immediate from the definition: If $\mathfrak{G}' := \bigcap_{P \in W} \mathfrak{G}(P)$, then $\text{Proj}(S/\mathfrak{G}')$ and $\text{Proj}(S/\mathfrak{G}(W))$ have the same support W , and

$$\mathfrak{G}'_Q = \mathfrak{G}(Q)_Q = \mathcal{C}_{\mathcal{O}_{\Gamma, Q}} = \mathfrak{G}(W)_Q$$

for all $Q \in W$, hence $\mathfrak{G}(W) = \mathfrak{G}'$. □

Proposition 5.4 yields Algorithm 3.

Algorithm 3 Gorenstein adjoint ideal, local to global

Input: A curve $\Gamma \subset \mathbb{P}_k^r$ as in Notation 5.1.

Output: The Gorenstein adjoint ideal \mathfrak{G} of Γ .

- 1: Compute $\text{Sing}(\Gamma) = \{P_1, \dots, P_s\}$.
 - 2: Apply Algorithm 4 in Section 6 below to compute $\mathfrak{G}(P_i)$ for all i .
 - 3: **return** $\bigcap_{i=1}^s \mathfrak{G}(P_i)$.
-

Remark 5.5. It is clear from Proposition 5.4 that we may choose any partition $\text{Sing}(\Gamma) = \bigcup_{i=1}^s W_i$ of $\text{Sing}(\Gamma)$ and have

$$\mathfrak{G} = \bigcap_{i=1}^s \mathfrak{G}(W_i).$$

This is useful in that for some subsets W_i , specialized approaches or a priori knowledge may ease the computation of $\mathfrak{G}(W_i)$. In Section 7, we will present some ideas in this direction for plane curves.

6. COMPUTING LOCAL ADJOINT IDEALS

In this section, we modify Algorithm 2 so that it computes the local Gorenstein adjoint ideal at a point P from a minimal local contribution at P via ideal quotients.

We consider a curve $\Gamma \subset \mathbb{P}_k^r$ as in Notation 3.8 with affine part³ C and a point $P \in \text{Sing}(C)$. Let $\frac{1}{d}U$ be the minimal local contribution to $\overline{k[C]}$ at P ; so $U \subset k[C]$ is an ideal and $d \in U$ is nonzero.

Proposition 6.1. *With notation as above, and given $Q \in C$, we have*

$$(\langle d \rangle_{k[C]} : U)_Q = \begin{cases} \mathcal{C}_{\mathcal{O}_{C, Q}} & \text{if } Q = P, \\ \mathcal{O}_{C, Q} & \text{if } Q \neq P. \end{cases}$$

Proof. By the minimality assumption, we have

$$\left(\frac{1}{d}U \right)_Q = \begin{cases} \overline{\mathcal{O}_{C, Q}} & \text{if } Q = P, \\ \mathcal{O}_{C, Q} & \text{if } Q \neq P. \end{cases}$$

The claim follows since localization commutes with forming the conductor:

$$\left(\langle d \rangle_{k[C]} : U \right)_Q = \left(\mathcal{C}_{(\frac{1}{d}U) / k[C]} \right)_Q = \mathcal{C}_{(\frac{1}{d}U)_Q / k[C]_Q}.$$

□

³To cover all singular points of Γ , we may have to choose affine charts other than that considered in Notation 3.8.

Now, we argue as in the proof of Proposition 3.9: From Proposition 6.1 and Remark 5.3, it follows that $\langle d \rangle_{k[C]} : U$ coincides with the ideal obtained by dehomogenizing $\mathfrak{G}(P)$ with respect to X_0 and mapping the result to $k[C]$. Hence, since $\mathfrak{G}(P)$ is saturated, Algorithm 4 below indeed computes $\mathfrak{G}(P)$.

Algorithm 4 Local Gorenstein adjoint ideal from local contribution

Input: A curve $\Gamma \subset \mathbb{P}_k^r$ with affine part C as in Notation 3.8 and a point $P \in \text{Sing}(C)$.

Output: The local Gorenstein adjoint ideal $\mathfrak{G}(P)$ of C .

1: Compute polynomials $d, a_0, \dots, a_s \in k[X_1, \dots, X_r]$ such that the fractions $\frac{a_i(x_1, \dots, x_r)}{d(x_1, \dots, x_r)}$ generate the minimal local contribution to $\overline{k[C]}$ at P as a $k[C]$ -module.

2: Compute the ideal quotient

$$\mathcal{C} = (\langle d \rangle + I(C)) : (\langle a_0, \dots, a_s \rangle + I(C)) \subset k[X_1, \dots, X_r].$$

3: **return** the homogenization of \mathcal{C} with respect to X_0 .

Example 6.2. We compute the local Gorenstein adjoint ideals for the curve given in Example 2.10 with affine equation

$$X^5 - Y^2(1 - Y)^3 = 0.$$

For the A_4 -singularity P_1 , we found

$$d_1 = x^2 \quad \text{and} \quad U_1 = \langle x^2, y(y-1)^3 \rangle_{\mathbb{C}[C]},$$

so that

$$\mathfrak{G}(P_1) = \langle X^2, Y \rangle.$$

For the E_8 singularity P_2 , we observed that

$$d_2 = x^3 \quad \text{and} \quad U_2 = \langle x^3, x^2 y^2 (y-1), y^2 (y-1)^2 \rangle_{\mathbb{C}[C]},$$

leading to

$$\mathfrak{G}(P_2) = \langle X^3, X(Y-Z), (Y-Z)^2 \rangle.$$

Note that $\mathfrak{G}(P_1)$ and $\mathfrak{G}(P_2)$ are the ideals already obtained in Example 4.7.

7. IMPROVEMENTS TO THE LOCAL STRATEGY FOR PLANE CURVES

In this section, we focus on the case of a plane curve Γ with affine part $C = V(f)$ and $\text{Sing}(\Gamma) = \text{Sing}(C)$ as in Notation 4.2. *For simplicity of the presentation, we suppose throughout the section that our ground field $k = \mathbb{C}$.*

As explained in Section 5, the Gorenstein adjoint ideal \mathfrak{G} can be computed as the intersection of local Gorenstein ideals via a partition of $\text{Sing}(C)$. To begin with, consider the following partition:

$$\text{Sing}(C) = W_2 \cup W_3 \cup \dots \cup W_r \cup W', \quad (7.1)$$

where, for all i , W_i denotes the locus of ordinary i -fold points (ordinary multiple points of multiplicity i), and where W' collects the remaining singularities of C . In particular, W_2 is the set of nodes of C .

Lemma 7.1. *Let $P \in \text{Sing}(C)$, and let $\mathfrak{m}_P \subset k[X, Y]$ be the corresponding maximal ideal. If P is an ordinary i -fold point of C , then*

$$\mathfrak{G}(P) = \mathfrak{m}_P^{i-1}.$$

Proof. Since C is a plane curve and P is an ordinary i -fold point of C , the conductor $\mathcal{C}_{\mathcal{O}_{C,P}} = \mathfrak{m}_{C,P}^{i-1}$, where $\mathfrak{m}_{C,P}$ is the maximal ideal of $\mathcal{O}_{C,P}$ (see [Matlis 1970], [Greco and Valabrega 1979]). The result follows from the very definition of $\mathfrak{G}(P)$. \square

Applying the lemma to the partition (7.1), we get the intersection of ideals

$$\mathfrak{G} = I(W_2) \cap I(W_3)^2 \cap \cdots \cap I(W_r)^r \cap \mathfrak{G}(W'). \quad (7.2)$$

Hence, in the case where Γ is known to have ordinary multiple points as singularities only (that is, $W' = \emptyset$), we can compute \mathfrak{G} in a very efficient way by using Algorithm 5 below (see [Böhm 1999]).

Algorithm 5 Gorenstein adjoint ideal, ordinary multiple points only

Input: A plane curve Γ of degree n with defining polynomial F as in Notation 4.2 with only ordinary multiple points.

Output: The Gorenstein adjoint ideal \mathfrak{G} of Γ .

```

1:  $J_1 := \langle \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z} \rangle$  (the ideal defining  $\text{Sing}(\Gamma)$ )
2:  $i := 1$ 
3: while  $(J_i : \langle X, Y, Z \rangle^\infty) \neq \langle 1 \rangle$  do
4:    $i := i + 1$ 
5:    $J_i := \langle \frac{\partial^{j+l+m} F}{\partial X^j \partial Y^l \partial Z^m} \mid j+l+m = i, j, l, m \in \mathbb{N}_0 \rangle$ 
6:    $B := \langle X, Y, Z \rangle^{n-i}$ 
7:   while  $i > 0$  do
8:      $I_i := (J_{i-1} : B^\infty)$  (the ideal of the  $i$ -fold points of  $\Gamma$ )
9:      $B := ((B \cap I_i^{i-1}) : \langle X, Y, Z \rangle^\infty)$ 
10:     $i := i - 1$ 
11: return  $B$ 
```

In the general case, Equation (7.2) allows us to reduce the computation of \mathfrak{G} to the less involved task of computing $\mathfrak{G}(W')$ as soon as we detect the ordinary i -fold points. To begin with treating these, here is how to find the nodes:

Remark 7.2. We know how to find *all* singularities: $\text{Sing}(C)$ is given by the ideal

$$J = \left\langle f, \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y} \right\rangle.$$

By the Morse lemma (see [Milnor 1968]), a point $P \in \text{Sing}(C)$ is a node iff the Hessian matrix $\text{Hess}(f)$ formed by the second partial derivatives of f is non-degenerate at P . That is, P is a node iff

$$I(P) + \langle \det(\text{Hess}(f)) \rangle = k[X, Y].$$

This gives us a fast way of computing W_2 .

Carrying our efforts one step further, we discuss the local analysis of the singularities via invariants. This yields an efficient method not only for finding the delta invariant, but also for detecting the ordinary i -fold points, for each i :

Remark 7.3. Let $P \in \text{Sing}(C)$. After a translation, we may assume that $P = (0, 0)$ is the origin. Write m_P for the multiplicity and

$$\mu_P = \dim_k \left(k[[X, Y]] / \left\langle \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y} \right\rangle \right)$$

for the *Milnor number* of C at P . Then $m_P = \deg h_P$, where h_P is the lowest degree homogeneous summand of the Taylor expansion of f at P . Recall that μ_P can be computed via standard bases (see [Greuel and Pfister 2008]). Furthermore,

if the Newton polygon of f is non-degenerate (otherwise, successively blow up), the number of branches of f at P can be computed as

$$r_P = \sum_{j=1}^{s-1} \gcd \left(V_X^{(j+1)} - V_X^{(j)}, V_Y^{(j+1)} - V_Y^{(j)} \right),$$

where $V^{(1)}, \dots, V^{(s)}$ are the (ordered) vertices of the Newton polygon (and X and Y refer to the respective coordinates). This is immediate from [Brieskorn and Knörrer 1986, Section 8.4, Lemma 3]. The delta invariant of C at P is then obtained as

$$\delta_P = \frac{1}{2}(\mu_P + r_P - 1)$$

(see, for example, [Greuel et al. 2007, Chapter 1, Proposition 3.34]). Furthermore, P is an ordinary i -fold point iff h_P is square-free and $m_P = i$. Equivalently,

$$(m_P, r_P, \delta_P) = \left(i, i, \binom{i}{2} \right).$$

See [Greuel et al. 2007, Chapter 1, Proposition 3.33].

The local analysis of the singularities may be used to further refine our partition of $\text{Sing}(C)$. For example, singularities of type ADE can be identified as follows:

Remark 7.4. With notation as in Remark 7.3, the point $P = (0, 0) \in \text{Sing}(C)$ is

- (1) of type A_n , $n \geq 2$, iff $h_P = l_1^2$, with $l_1 \in k[X, Y]$ linear, and $\mu_P = n$,
- (2) of type D_n , $n \geq 4$, iff $h_P = l_1 l_2 l_3$ or $h_P = l_1^2 l_2$, with pairwise different linear polynomials $l_j \in k[X, Y]$, and $\mu_P = n$, and
- (3) of type E_n , $n = 6, 7, 8$, iff $h_P = l_1^3$, with $l_1 \in k[X, Y]$ linear, and $\mu_P = n$.

Here, in (2), h_P splits into three different linear factors iff P is of type D_4 . See, for example, [Greuel et al. 2007, Chapter 1, Theorems 2.48, 2.51, 2.54].

To describe the local Gorenstein adjoint ideal at a singularity of type A , D , or E , we use the following notation:

Notation 7.5. For any element $g \in k[[X, Y]]$, let $g_j = \text{taylor}(g, j) \in k[X, Y]$ be the Taylor expansion of g at $P = (0, 0)$ modulo $O(j+1)$.⁴

If C has a singularity of type A_n at $P = (0, 0)$, we may write f in the form $f = T^2 + W^{n+1}$, where $T, W \in k[[X, Y]]$ is a regular system of parameters. Let $s = \lfloor \frac{n+1}{2} \rfloor$ (the meaning of s will become clear in the proof of Lemma 7.6). We may compute the Taylor expansion $T_{s-1} \in k[X, Y]$ as follows. If n and thus s is equal to 1, set $T_0 = 0$. Otherwise, inductively solve f for T : Start by choosing a linear form $T_1 \in k[X, Y]$ such that $\text{taylor}(f, 2) = T_1^2$. Supposing that $1 < j < s-1$ and $T_j = T + O(j+1)$ has already been computed, write

$$\text{taylor}(f - T_j^2, j+2) = 2T_1 \cdot m,$$

with $m \in k[X, Y]$ homogeneous of degree $j+1$, and set $T_{j+1} = T_j + m$.

Lemma 7.6. *Let C have a singularity of type A_n , $n \geq 1$, at $P = (0, 0)$. Set $s = \lfloor \frac{n+1}{2} \rfloor$, and let T_{s-1} be defined as above. Then $\mathfrak{G}(P)$ is the homogenization of*

$$\langle X^s, T_{s-1}, Y^s \rangle \subset k[X, Y]$$

with respect to Z .

Proof. The case $n = 1$ is clear, so we may suppose $n \geq 2$. If $\mathfrak{G}' = \langle X^s, T_{s-1}, Y^s \rangle \subset k[X, Y]$, then $\mathfrak{G}'_Q = \mathcal{O}_{C,Q}$ for all $Q \in C \setminus \{P\}$, so it suffices to show that $\mathfrak{G}'_P = \mathcal{C}_B$, where $B = \mathcal{O}_{C,P}$. For this, we pass to the completion

$$\widehat{B} = k[[x, y]] = k[[X, Y]] / \langle f(X, Y) \rangle,$$

⁴The notation $O(m)$ stands for terms of degree $\geq m$.

and consider the isomorphism

$$A = k[[t, w]] = k[[T, W]] / \langle T^2 + W^{n+1} \rangle \rightarrow \widehat{B}, t \mapsto T(x, y), w \mapsto W(x, y).$$

An analysis of the normalization algorithm applied to A shows that

$$\overline{A} = \sum_{i=0}^{n-s} k[[t]] \cdot w^i + \sum_{i=n-s+1}^n k[[t]] \cdot \frac{w^i}{t},$$

and that it takes $s = \lfloor \frac{n+1}{2} \rfloor$ steps to reach \overline{A} (see [Böhm et al. 2014, Sect. 4]). Hence,

$$\mathcal{C}_A = \langle t, w^s \rangle_A, \text{ so that } \mathcal{C}_{\widehat{B}} = \langle T(x, y), W(x, y)^s \rangle_{\widehat{B}}.$$

Working in $k[[X, Y]]$, we write

$$T = aX + bY \quad \text{and} \quad W = cX + dY,$$

with $a, b, c, d \in k[[X, Y]]$ and such that $ad - bc$ is a unit in $k[[X, Y]]$. Since $\langle X, Y \rangle = \langle T, W \rangle$, it follows that $\langle X, Y \rangle^s = \langle T, W \rangle^s \subset \langle T, W^s \rangle$. Since $\langle X, Y \rangle = \langle X, T \rangle$ or $\langle X, Y \rangle = \langle T, Y \rangle$, we have $W^s \in \langle X, Y \rangle^s \subset \langle X^s, T, Y^s \rangle$. We conclude that

$$\langle X^s, T, Y^s \rangle = \langle T, W^s \rangle.$$

If $s > 1$, then $\langle X, Y \rangle = \langle X, T_{s-1} \rangle$ or $\langle X, Y \rangle = \langle T_{s-1}, Y \rangle$, hence, for any s , we have $\langle X, Y \rangle^s \subset \langle X^s, T_{s-1}, Y^s \rangle$. We conclude that

$$\langle X^s, T_{s-1}, Y^s \rangle = \langle X^s, T, Y^s \rangle.$$

Now recall that B is an excellent ring, which implies that $\overline{\widehat{B}} = \widehat{\overline{B}}$ (see, for example, [Böhm et al. 2014, Sect. 1]). It follows that

$$\mathcal{C}_{\widehat{B}} = \text{Hom}_{\widehat{B}}(\widehat{\overline{B}}, B) = \text{Hom}_B(\overline{B}, B) \otimes_B \widehat{B} = \mathcal{C}_B \otimes_B \widehat{B}. \quad (7.3)$$

Since completion is faithfully flat in the case considered here, we conclude that

$$\mathcal{C}_B = \langle x^s, T_{s-1}(x, y), y^s \rangle_B.$$

□

Remark 7.7. In particular, if P is a cusp, then $\mathfrak{G}(P) = \langle X, Y \rangle$. So, in (7.2), nodes and cusps may be treated simultaneously.

If C has a singularity of type D_n at $P = (0, 0)$, we may write f in the form $f = W \cdot (T^2 + W^{n-2})$, where $T, W \in k[[X, Y]]$ is a regular system of parameters. Let $s = \lfloor \frac{n}{2} \rfloor$. We may compute the Taylor expansion $T_{s-2} \in k[X, Y]$ as follows. If $n = 4$, set $T_0 = 0$. If $n \geq 5$, choose linear forms $T_1, W_1 \in k[X, Y]$ such that $\text{taylor}(f, 3) = T_1^2 \cdot W_1$. For $j \leq s - 2$, determine $W_j = W + O(j + 1)$ as the Puiseux expansion up to order j of f corresponding to W_1 . Supposing that $1 < j < s - 2$ and $T_j = T + O(j + 1)$ has already been computed, write

$$\text{taylor}(f - T_j^2 \cdot W_{j+1}, j + 3) = 2Z_1 \cdot W_1 \cdot m,$$

with $m \in k[X, Y]$ homogeneous of degree $j + 1$, and set $T_{j+1} = T_j + m$.

Lemma 7.8. *Let C have a singularity of type D_n , $n \geq 4$, at $P = (0, 0)$. Set $s = \lfloor \frac{n}{2} \rfloor$, and let T_{s-2} be defined as above. Then $\mathfrak{G}(P)$ is the homogenization of*

$$\langle X, Y \rangle \cdot \langle X^{s-1}, T_{s-2}, Y^{s-1} \rangle \subset k[X, Y]$$

with respect to Z .

Proof. We have an isomorphism

$$A \rightarrow \widehat{B}, q \mapsto T(x, y), w \mapsto W(x, y),$$

where $B = \mathcal{O}_{C,P}$ and

$$A = k[[t, w]] = k[[T, W]] / \langle W \cdot (T^2 + W^{n-2}) \rangle.$$

This time, the normalization is

$$\overline{A} = \sum_{i=0}^{n-2-s} k[[t]] \cdot w^i + \sum_{i=n-1-s}^{n-3} k[[t]] \cdot \frac{w^i}{t} + k[[t]] \cdot \frac{w^{n-2}}{t^2},$$

and it takes $s = \lfloor \frac{n}{2} \rfloor$ steps to reach \overline{A} (see again [Böhm et al. 2014, Sect. 4]). Hence,

$$\mathcal{C}_A = \langle t^2, tw, w^s \rangle.$$

Write

$$T = aX + bY \quad W = cX + dY$$

with $a, b, c, d \in k[[X, Y]]$ and such that $ad - bc$ is a unit in $k[[X, Y]]$. Since $\langle X, Y \rangle = \langle T, W \rangle$, we have $\langle XT, YT \rangle = \langle T^2, TW \rangle$ and $\langle X, Y \rangle^s = \langle T, W \rangle^s \subset \langle T^2, TW, W^s \rangle$, hence

$$\langle X, Y \rangle \cdot \langle X^{s-1}, T, Y^{s-1} \rangle \subset \langle T^2, TW, W^s \rangle.$$

For the other inclusion, observe that $\langle X, Y \rangle = \langle X, T \rangle$ or $\langle X, Y \rangle = \langle T, Y \rangle$, so it follows that $\langle X, Y \rangle^{s-1} \subset \langle X^{s-1}, T, Y^{s-1} \rangle$, hence

$$W^s \in \langle X, Y \rangle^s \subset \langle X, Y \rangle \cdot \langle X^{s-1}, T, Y^{s-1} \rangle.$$

If $s > 2$, then $\langle X, Y \rangle = \langle X, T_{s-2} \rangle$ or $\langle X, Y \rangle = \langle T_{s-2}, Y \rangle$, hence, for any s , we have $\langle X, Y \rangle^{s-1} \subset \langle X^{s-1}, T_{s-2}, Y^{s-1} \rangle$. We conclude that

$$\langle X^{s-1}, T_{s-2}, Y^{s-1} \rangle = \langle X^{s-1}, T, Y^{s-1} \rangle.$$

To summarize,

$$\langle T^2, TW, W^s \rangle = \langle X, Y \rangle \cdot \langle X^{s-1}, T, Y^{s-1} \rangle = \langle X, Y \rangle \cdot \langle X^{s-1}, T_{s-2}, Y^{s-1} \rangle,$$

hence

$$\mathcal{C}_{\widehat{B}} = \langle x, y \rangle \cdot \langle x^{s-1}, T_{s-2}(x, y), y^{s-1} \rangle \subset \widehat{B}.$$

Then the claim follows as before. \square

Lemma 7.9. *Let C have a singularity of type E_n , $n = 6, 7, 8$, at $P = (0, 0)$. Set $s = \lfloor \frac{n-1}{2} \rfloor$, and let l_1 be as in Remark 7.4. Then $\mathfrak{G}(P)$ is the homogenization of*

$$\langle X, Y \rangle \cdot \langle X^{s-1}, l_1, Y^{s-1} \rangle \subset k[X, Y]$$

with respect to Z .

Proof. Depending on $n \in \{6, 7, 8\}$, we have an isomorphism

$$A \rightarrow \widehat{B}, q \mapsto T(x, y), w \mapsto W(x, y),$$

where $B = \mathcal{O}_{C,P}$ and

$$A = k[[t, w]] = k[[T, W]] / \langle T^3 + W^4 \rangle,$$

$$A = k[[t, w]] = k[[T, W]] / \langle T (T^2 + W^3) \rangle,$$

$$A = k[[t, w]] = k[[T, W]] / \langle T^3 + W^5 \rangle,$$

respectively. In each case, by [Böhm et al. 2014, Sect. 4],

$$\overline{A} = k[[w]] \cdot 1 + k[[w]] \cdot \frac{t}{w} + k[[w]] \cdot \frac{t^2}{w^s},$$

which implies that

$$\mathcal{C}_A = \langle t^2, tw, w^s \rangle.$$

The same argument as in the proof of Lemma 7.8 shows that

$$\mathcal{C}_{\widehat{B}} = \langle x, y \rangle \cdot \langle x^{s-1}, T_{s-2}(x, y), y^{s-1} \rangle \subset \widehat{B},$$

and the claim follows as before. Note that $T_{s-2} = 0$ if $s = 2$, and $T_{s-2} = l_1$ if $s = 3$. \square

In principle, we could pursue a similar strategy for all singularities classified by Arnold in [Arnold et al. 1995]. However, in [Böhm et al. 2015a], we give an algorithm which, for plane curves in characteristic zero, allows us to compute the local contributions to the normalization for a broad class of singularities in a direct way. Combining the approach of Section 6 with this algorithm or with modular techniques and normalization as described in Section 8 below, we already get a very efficient algorithm for computing \mathfrak{G} .

Remark 7.10. For the local analysis of the singularities, we temporarily may have to leave k .

8. PARALLEL COMPUTATION AND MODULAR TECHNIQUES

Algorithm 3 is parallel in nature since the computations of the local adjoint ideals do not depend on each other. In this section, in the case where the given curve is defined over \mathbb{Q} , we describe a modular way of parallelizing Algorithm 3 even further. One possible approach is to replace the computations of the Gröbner bases involved, the computation of the (minimal) associated primes in the singular locus, and the computations yielding the normalizations by their modular variants as introduced by [Arnold 2003], [Idrees et al. 2011], and [Böhm et al. 2012a]. These variants are either probabilistic or require expensive tests to verify the results at the end. In order to reduce the number and complexity of the verification tests, we provide a direct modularization for the adjoint ideal algorithm. The approach we propose requires only the verification of the final result: We give efficient conditions for checking whether the result obtained is indeed the Gorenstein adjoint ideal.

Our approach relies on the general scheme for modular computations presented in [Böhm et al. 2012d]. This scheme is based on error tolerant rational reconstruction (see Remark 8.6 below) and can handle *bad primes*⁵ of various types, provided there are only finitely many such primes. Referring to [Böhm et al. 2012d] for details, we will now outline the main ideas behind the scheme.

Fix a global monomial ordering $>$ on the monoid of monomials in the variables $X = \{X_0, \dots, X_r\}$. Consider the polynomial rings $R = \mathbb{Q}[X]$ and, given an integer $N \geq 2$, $R_N = (\mathbb{Z}/N\mathbb{Z})[X]$. If $H \subset R$ or $H \subset R_N$ is a Gröbner basis, then denote by $\text{LM}(H) := \{\text{LM}(f) \mid f \in H\}$ its set of leading monomials.

If $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$ and $\gcd(b, N) = 1$, set $(\frac{a}{b})_N := (a + N\mathbb{Z})(b + N\mathbb{Z})^{-1} \in \mathbb{Z}/N\mathbb{Z}$. If $f \in R$ is a polynomial such that N is coprime to any denominator of a coefficient of f , then its *reduction modulo N* is the polynomial $f_N \in R_N$ obtained by mapping each coefficient x of f to x_N . If $H = \{h_1, \dots, h_t\} \subset R$ is a Gröbner basis such that N is coprime to any denominator in any h_i , set $H_N = \{(h_1)_N, \dots, (h_t)_N\}$. If $J \subset R$ is an ideal, we write

$$J_0 = J \cap \mathbb{Z}[X] \quad \text{and} \quad J_N = \langle f_N \mid f \in J_0 \rangle \subset R_N,$$

and call J_N the *reduction of J modulo N* . We also write $(R/J)_N = R_N/J_N$.

Based on this notation, we fix the following setup for the rest of this section:

Notation 8.1. Let $\Gamma \subset \mathbb{P}_{\mathbb{Q}}^r$ be a curve of degree n . As before, suppose that Γ is integral and non-degenerate. Denote by $I(\Gamma)$ the ideal of Γ in R , and by $G(0) \subset R$ the reduced Gröbner basis of $\mathfrak{G}(\Gamma)$. If p is a prime such that $I(\Gamma)_p$ is radical and

⁵In our context, a prime p is *bad* if Algorithm 3 applied to the modulo p values of the input over the rationals does not return the reduction of the characteristic zero result.

defines an integral, non-degenerate curve in $\mathbb{P}_{\mathbb{F}_p}^r$, then write Γ_p for this curve and $G(p) \subset R_p$ for the reduced Gröbner basis of $\mathfrak{G}(\Gamma_p)$.

Remark 8.2. Given p , the ideal $I(\Gamma)_p$ can be found using Gröbner bases over \mathbb{Z} (see [Adams and Loustau 1994, Cor. 4.4.5] and [Arnold 2003, Lem. 6.1]). We will make use of this in the final verification test. With regard to the other steps of our algorithm (in particular, in a randomized version of the algorithm obtained by omitting the verification test), we can proceed in the following, more efficient way: Let $\{f_1, \dots, f_r\}$ be the reduced Gröbner basis of $I(\Gamma)$. Reject p if one of the $(f_i)_p$ is not defined (there are only finitely many such primes p). Otherwise, realize $I(\Gamma)_p$ via the equality

$$I(\Gamma)_p = \langle (f_1)_p, \dots, (f_r)_p \rangle \subset R_p, \quad (8.1)$$

which holds true for all but finitely many primes p . These finitely many bad primes will not influence the lift if we apply error tolerant rational reconstruction as described in Remark 8.6 below.

Remark 8.3. There are only finitely many primes p for which the desired conditions on $I(\Gamma)_p$ in Notation 8.1 are not satisfied. Since these conditions can be checked using polynomial factorization and Gröbner bases, we may simply reject such a bad prime if we encounter it in our modular algorithm. Hence, we will ignore these bad primes in the following discussion. In particular, we will assume that the Gröbner bases $G(p)$ exists for all primes p .

The basic idea of the modular adjoint ideal algorithm can then be described as follows: First, choose a set of primes \mathcal{P} and compute $G(p)$ for each $p \in \mathcal{P}$. Second, lift the $G(p)$ coefficientwise to a set of polynomials $G \subset R$. Provided that $\mathfrak{G}(\Gamma)_p = \mathfrak{G}(\Gamma_p)$ for each $p \in \mathcal{P}$, we then expect that G is a Gröbner basis which coincides with our target Gröbner basis $G(0)$.

The lifting process consists of two steps. First, use Chinese remaindering to lift the $G(p) \subset R_p$ to a set of polynomials $G(N) \subset R_N$, with $N := \prod_{p \in \mathcal{P}} p$. Second, compute a set of polynomials $G \subset R$ by lifting the coefficients occurring in $G(N)$ to rational coefficients. Here, to identify Gröbner basis elements corresponding to each other, we require that $\text{LM}(G(p)) = \text{LM}(G(q))$ for all $p, q \in \mathcal{P}$. This leads to condition (L2) in the definition below:

Definition 8.4. With notation as above, a prime p is called *lucky* if:

- (L1) $\mathfrak{G}(\Gamma)_p = \mathfrak{G}(\Gamma_p)$ and
- (L2) $\text{LM}(G(0)) = \text{LM}(G(p))$.

Otherwise p is called *unlucky*.

Lemma 8.5. *All but finitely many primes are lucky.*

Proof. As is clear from the proof of [Böhm et al. 2012d, Lemma 5.5], it is enough to show that condition (L1) is true for all but finitely many primes. For this, we may assume that both Γ and Γ_p do not have any singularities at $X_0 = 0$. Let C be the affine part of Γ . Write $A = \mathbb{Q}[X_1, \dots, X_r]/I(C)$. As shown in [Böhm et al. 2012a], $(\bar{A})_p = \bar{A}_p$ for all but finitely many primes p . So if we write $\bar{A} = \frac{1}{d}U$, with an ideal $U \subset A$ and an element $0 \neq d \in A$, and $\bar{A}_p = \frac{1}{d(p)}U(p)$, with $U(p) \subset A_p$ and $d_p \in A_p$, then

$$(d_p : U_p) = (d(p) : U(p))$$

for all but finitely many primes p . Computing an ideal quotient amounts to a Gröbner basis computation. Hence, as pointed out in [Böhm et al. 2012d, Remark 5.3],

$$(d : U)_p = (d_p : U_p)$$

for all but finitely many primes p . The result follows, thus, from Propositions 3.9 and 4.8. \square

When performing our modular algorithm, condition (L1) can only be checked a posteriori: We compute $G(p)$ and, thus, $\mathfrak{S}(\Gamma_p)$ on our way, but $\mathfrak{S}(\Gamma)_p$ is only known to us after $G(0)$ and, thus, $\mathfrak{S}(\Gamma)$ has been computed. This is not a problem, however, since the finitely many primes where $\mathfrak{S}(\Gamma)_p \neq \mathfrak{S}(\Gamma_p)$ will not influence the final result if we apply error tolerant rational reconstruction and the set \mathcal{P} is large enough:

Remark 8.6. Let N' and M be integers with $\gcd(N', M) = 1$, let $N = N' \cdot M$, and let $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = \gcd(N', b) = 1$. Set $r_1 := (\frac{a}{b})_{N'} \in \mathbb{Z}/N'\mathbb{Z}$, let $r_2 \in \mathbb{Z}/M\mathbb{Z}$ be arbitrary, and denote by r the image of (r_1, r_2) under the isomorphism

$$\mathbb{Z}/N'\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z}.$$

Lifting r to a rational number by Gaussian reduction, starting from $(a_0, b_0) = (N'M, 0)$ and $(a_1, b_1) = (r, 1)$, we create the sequence (a_i, b_i) obtained by

$$(a_{i+2}, b_{i+2}) = (a_i, b_i) - q_i(a_{i+1}, b_{i+1}),$$

with

$$q_i = \left\lfloor \frac{\langle (a_i, b_i), (a_{i+1}, b_{i+1}) \rangle}{\|(a_{i+1}, b_{i+1})\|^2} \right\rfloor.$$

Computing this sequence until $\|(a_{i+2}, b_{i+2})\| \geq \|(a_{i+1}, b_{i+1})\|$, we return **false** if $\|(a_{i+1}, b_{i+1})\|^2 \geq N$, and $\frac{a_{i+1}}{b_{i+1}}$, otherwise. By [Böhm et al. 2012d, Lemma 4.3], this algorithm will return $\frac{a_{i+1}}{b_{i+1}} = \frac{a}{b}$, provided that N is large enough and $M \ll N'$. More precisely, we ask that $N' > (a^2 + b^2) \cdot M$.

Definition 8.7. If \mathcal{P} is a finite set of primes, set

$$N' = \prod_{p \in \mathcal{P} \text{ lucky}} p \quad \text{and} \quad M = \prod_{p \in \mathcal{P} \text{ unlucky}} p.$$

Then \mathcal{P} is called *sufficiently large* if

$$N' > (a^2 + b^2) \cdot M$$

for all coefficients $\frac{a}{b}$ of polynomials in $G(0)$ (assume $\gcd(a, b) = 1$).

Lemma 8.8. *If \mathcal{P} is a sufficiently large set of primes satisfying condition (L2), then the reduced Gröbner bases $G(p)$, $p \in \mathcal{P}$, lift to the reduced Gröbner basis $G(0)$.*

Proof. See [Böhm et al. 2012d, Lemma 5.6]. \square

From a theoretical point of view, Lemma 8.5 guarantees that a sufficiently large set \mathcal{P} of primes satisfying condition (L2) exists. From a practical point of view, however, (L2) can only be checked a posteriori. Nevertheless, in order to be able to identify Gröbner basis elements in the lifting process, we have to restrict to a set of primes p which all have the same associated set of lead monomials $\text{LM}(G(p))$. Hence, taking Lemma 8.5 into account, we proceed along the following lines: First, fix an integer $t \geq 1$ and choose a set of t primes \mathcal{P} at random. Second, compute $\mathcal{GP} = \{G(p) \mid p \in \mathcal{P}\}$ and use a majority vote with respect to (L2):

DELETEBYMAJORITYVOTE: *Define an equivalence relation on \mathcal{P} by setting $p \sim q :\iff \text{LM}(G(p)) = \text{LM}(G(q))$. Then replace \mathcal{P} by the equivalence class of largest cardinality,⁶ and change \mathcal{GP} accordingly.*

Now, all $G(p)$, $p \in \mathcal{P}$, have the same set of leading monomials. Hence, we can apply the rational reconstruction algorithm to the coefficients of the Gröbner bases in \mathcal{GP} . If this algorithm returns **false** at some point, we enlarge the set \mathcal{P} by t primes not used so far, and repeat the whole process. Otherwise, the lifting yields a set of

⁶We have to use a weighted cardinality count: when enlarging \mathcal{P} , the total weight of the elements already present must be strictly smaller than the total weight of the new elements. Otherwise, though highly unlikely in practical terms, it may happen that only unlucky primes are accumulated.

polynomials $G \in R$. Furthermore, if \mathcal{P} is sufficiently large, all primes in \mathcal{P} satisfy condition (L2). Since we cannot check, however, whether \mathcal{P} is sufficiently large, a final verification step is needed. Since this may be expensive, especially if $G \neq G(0)$, we first perform a test in positive characteristic:

PTEST: *Randomly choose a prime $p \notin \mathcal{P}$ which does not divide the numerator or denominator of any coefficient occurring in a polynomial in G . Return **true** if $G_p = G(p)$, and **false** otherwise.*

If PTEST returns **false**, then \mathcal{P} is not sufficiently large (or the extra prime chosen in PTEST is bad). In this case, we enlarge \mathcal{P} as above and repeat the process. If PTEST returns **true**, however, then most likely $G = G(0)$. In this case, we verify the result over the rationals as described below. If the verification fails, we again enlarge \mathcal{P} and repeat the process.

We now discuss the verification. We write $I = \langle G \rangle_R$ for the lifted modular result and $\mathfrak{G} = \mathfrak{G}(\Gamma) \subset R$ for the correct result. After checking that G is indeed a Gröbner basis and I is saturated (henceforth, this will be assumed), we apply the following results.

Lemma 8.9. *With notation as above, the ideal I is equal to the Gorenstein adjoint ideal \mathfrak{G} of Γ iff*

- (1) $I(\Gamma) \subsetneq I$,
- (2) $\deg \Delta(I) = \deg I + \delta(\Gamma)$, and
- (3) $\deg I = \deg \mathfrak{G}$.

Proof. If $I = \mathfrak{G}$, then I satisfies (1), (2), and (3). Conversely, by Lemma 3.13, conditions (1) and (2) imply that I is an adjoint ideal of Γ . In this case, since \mathfrak{G} is the largest such ideal, we have $I \subset \mathfrak{G}$. But then $I = \mathfrak{G}$ by (3). \square

It is clear how to check condition (1). In what follows, we describe a method for checking (2) which, in particular, provides a way of finding $\deg \Delta(\mathfrak{G})$. This will allow us to check (3) via the formula $\deg \mathfrak{G} = \deg \Delta(\mathfrak{G}) - \delta(\Gamma)$.

If k is any field, and A is any reduced Noetherian k -algebra, the *delta invariant* of A is defined to be

$$\delta_k(A) = \dim_k \overline{A}/A.$$

Proposition 8.10. *Let B be a ring, and let A be a B -algebra with the following properties:*

- (1) (B, \mathfrak{m}) is a normal local ring with perfect residue class field k .
- (2) $B \rightarrow \widehat{B}$ is flat, and for all $\mathfrak{p} \in \text{Spec}(B)$ such that $\mathfrak{p}\widehat{B} \neq \widehat{B}$, the ring $\widehat{B} \otimes_{B/\mathfrak{p}} B/\mathfrak{p}$ is geometrically normal.
- (3) A is a formally equidimensional Nagata ring.
- (4) A is a flat B -algebra, $\mathfrak{m}A$ is contained in every maximal ideal of A , $A/\mathfrak{m}A$ is reduced, and $\delta_k(A/\mathfrak{m}A) < \infty$.
- (5) \overline{A}/A is a finite B -module.
- (6) The unique map $\overline{A}/\mathfrak{m}\overline{A} \rightarrow \overline{A/\mathfrak{m}A}$ factorizing the normalization map $A/\mathfrak{m}A \rightarrow \overline{A/\mathfrak{m}A}$ as

$$A/\mathfrak{m}A \rightarrow \overline{A}/\mathfrak{m}\overline{A} \rightarrow \overline{A/\mathfrak{m}A}$$

is injective.

Then

$$\delta_{Q(B)}(A \otimes_B Q(B)) \leq \delta_k(A/\mathfrak{m}A).$$

Proof. See [Lipman 2006, Prop. 2.2.1(i)] for the factorization in (6) and [Lipman 2006, Prop. 3.3] for the proof of the proposition. \square

Corollary 8.11. *In the setting of Notation 8.1, given a prime p , we have*

$$\delta(\Gamma) \leq \delta(\Gamma_p).$$

Proof. Let $X' = \{X_1, \dots, X_r\}$. We may assume that Γ has no singularities at $X_0 = 0$. As before, let C be the affine part of Γ . Then $J := I(C)_0 \subset \mathbb{Z}[X']$ is a prime ideal of height $n-1$, $\langle p, J \rangle$ is a prime ideal, and $J \cap \mathbb{Z} = \langle 0 \rangle$. The claim follows by applying Proposition 8.10 to $(B, \mathfrak{m}) = (\mathbb{Z}_{\langle p \rangle}, \langle p \rangle)$ and $A = \mathbb{Z}_{\langle p \rangle}[X']/J \mathbb{Z}_{\langle p \rangle}[X']$ since, then, $A \otimes_B \mathbb{Q}(B) = \mathbb{Q}[X']/I(C)$ and $A/\mathfrak{m}A = \mathbb{F}_p[X']/I(C)_p$, and conditions (1) through (6) of the proposition are satisfied. Indeed, this is clear for (1), and (2) holds since B is excellent. Moreover, we have (3) since A is of finite type over B and $J \mathbb{Z}_{\langle p \rangle}[X']$ is a prime ideal. Condition (4) follows since A is a torsion free B -module, $\langle p, J \rangle$ is a prime ideal, and $\text{Spec}(A/\mathfrak{m}A)$ is a curve. We obtain (5) since A/\mathcal{C}_A is a finite B -module and $\overline{A}/\mathcal{C}_A$ is a finite A/\mathcal{C}_A -module. Condition (6) follows from Lemma 8.13 below which gives us a canonical map

$$\overline{A} \rightarrow \overline{A/\mathfrak{m}A}, \alpha = \frac{\overline{a}}{\overline{b}} \mapsto \frac{a \bmod \langle p, J \rangle}{b \bmod \langle p, J \rangle},$$

where $\overline{a}, \overline{b}$ are the images of $a, b \in \mathbb{Z}_{\langle p \rangle}[X']$ in A , and $b \notin \langle p, J \rangle$. Since $\alpha = \frac{\overline{a}}{\overline{b}}$ is in the kernel of this map iff $a \in \langle p, J \rangle$, we get an injective map $\overline{A/\mathfrak{m}A} \rightarrow \overline{A/\mathfrak{m}A}$ which factors the normalization map as desired. \square

Before deriving Lemma 8.13, we illustrate condition (6) by an example.

Example 8.12. Let $(B, \mathfrak{m}) = (\mathbb{Z}_{\langle 3 \rangle}, \langle 3 \rangle)$ and $A = \mathbb{Z}_{\langle 3 \rangle}[X, Y]/\langle X^3 + Y^3 + Y^5 \rangle$. Then $\overline{A/\mathfrak{m}A} = \left\langle 1, \frac{x}{y}, \frac{(x+y)^2}{y^3} \right\rangle_{A/\mathfrak{m}A}$ and $\overline{A} = \left\langle 1, \frac{x}{y}, \frac{x^2}{y^2} \right\rangle_A$. We compute $\delta_{\mathbb{Q}}(A \otimes_B \mathbb{Q}) = 3$ and $\delta_{\mathbb{F}_p}(A/\mathfrak{m}A) = 4$, and find that

$$\overline{A/\mathfrak{m}A} = \left\langle 1, \frac{x}{y}, \frac{x^2}{y^2} \right\rangle_{A/\mathfrak{m}A} \subsetneq \left\langle 1, \frac{x}{y}, \frac{(x+y)^2}{y^3} \right\rangle_{A/\mathfrak{m}A} = \overline{A/\mathfrak{m}A}.$$

Lemma 8.13. *With the notation of the proof of Corollary 8.11, for any $\alpha \in \overline{A}$ there exist $a, b \in \mathbb{Z}[X']$ with $b \notin \langle p, J \rangle$ such that $\alpha = \frac{\overline{a}}{\overline{b}}$.*

Proof. For $\alpha \in \overline{A} \subset \mathbb{Q}(A) = \mathbb{Q}(\mathbb{Z}[X']/J)$, there are $a, b \in \mathbb{Z}[X']$ with $b \notin J$ and $\alpha = \frac{a \bmod J}{b \bmod J}$, and there are $a_0, \dots, a_{m-1} \in \mathbb{Z}[X']$ and $d \in \mathbb{Z}$ with $p \nmid d$ such that

$$\alpha^m + \frac{a_{m-1} \bmod J}{d} \alpha^{m-1} + \dots + \frac{a_0 \bmod J}{d} = 0,$$

that is, $d \cdot a^m + a_{m-1} \cdot ba^{m-1} + \dots + a_0 \cdot b^m \in J$.

If $b \in \langle p, J \rangle$, then $d \cdot a^m \in \langle p, J \rangle$, hence, since J is radical, $a \in \langle p, J \rangle$. Then $a = pa_1 + c_1$ and $b = pb_1 + d_1$ with $a_1, b_1 \in \mathbb{Z}[X']$ and $c_1, d_1 \in J$. If $b_1 \in \langle p, J \rangle$, we can iterate the process. Inductively, we obtain $a_s, b_s \in \mathbb{Z}[X']$ and $c_s, d_s \in J$ with $a = p^s a_s + c_s$ and $b = p^s b_s + d_s$. If $b_s \in \langle p, J \rangle$ for all s , then $b \in \bigcap_s \langle p^s, J \rangle = J$, a contradiction. Otherwise there is an s with $b_s \notin \langle p, J \rangle$. Then

$$\alpha = \frac{a \bmod J}{b \bmod J} = \frac{p^s a_s \bmod J}{p^s b_s \bmod J} = \frac{a_s \bmod J}{b_s \bmod J}.$$

\square

In the following, we write again $\pi: \overline{\Gamma} \rightarrow \Gamma$ for the normalization map, and denote by M the vanishing ideal of $\text{Sing}(\Gamma)$ in R . Consider a homogeneous polynomial $g \in I = \langle G \rangle_R$ not contained in $I(\Gamma)$, and let m be its degree. Let $\text{div}(g)$ be the divisor cut out by π^*g on $\overline{\Gamma}$, let $D(g) = \text{div}(g) - \Delta(I)$ be the corresponding divisor in $|mH - \Delta(I)|$, and let $d(g) = \deg D(g)$. Furthermore, write $\tilde{d}(g)$ for the degree of the part of $D(g)$ away from $\text{Sing}(\Gamma)$. Then $\tilde{d}(g) \leq d(g)$, and $\tilde{d}(g)$ can be computed as

$$\tilde{d}(g) = \deg((I(\Gamma) + \langle g \rangle) : M^\infty)$$

provided that $I : M^\infty = \langle 1 \rangle$, what we will henceforth assume (in Algorithm 6 below, if this condition is not fulfilled, we enlarge our set of primes).

Theorem 8.14. *Let $I = \langle G \rangle_R$ be as above, and let p be a prime number. Suppose:*

- (1) $\text{LM}(I(\Gamma_p)) = \text{LM}(I(\Gamma))$,
- (2) $G(p)$ is a Gröbner basis of an adjoint ideal of Γ_p ,
- (3) $G_p = G(p)$,
- (4) $\tilde{d}(g_p) = \deg(\Gamma) \cdot m - \deg(\langle G(p) \rangle_{R_p}) - \delta(\Gamma)$, and
- (5) m is large enough to ensure that $|mH - \Delta(I)|$ is nonspecial.

Then

$$\deg \Delta(I) = \deg(\Gamma) \cdot m - \tilde{d}(g_p).$$

Furthermore, $\deg \Delta(I) = \deg \Delta(I_p)$, and I is an adjoint ideal of Γ .

Remark 8.15. To apply the theorem in the setup above, note: Condition (1) can easily be tested. Furthermore, (2) and (3) are satisfied by the construction of G . Since we know how to compute $\delta(\Gamma)$, condition (4) can be tested. With respect to (5), we will comment on how to choose m in Lemma 8.17 below.

Proof of the theorem. By (1), $\deg(\Gamma_p) = \deg(\Gamma)$ and $p_a(\Gamma_p) = p_a(\Gamma)$. First note, that by (3)

$$I_p = \langle G \rangle_{R_p} = \langle G(p) \rangle_{R_p}$$

and, as G is assumed to be a Gröbner basis,

$$\deg(\langle G \rangle_R) = \deg(\langle G(p) \rangle_{R_p}). \quad (8.2)$$

By Corollary 8.11, we have $\delta(\Gamma) \leq \delta(\Gamma_p)$. Hence

$$\begin{aligned} \tilde{d}(g_p) &\leq d(g_p) = \deg(\Gamma_p) \cdot m - \deg \Delta(I_p) \\ &= \deg(\Gamma) \cdot m - \deg(I_p) - \delta(\Gamma_p) \\ &\leq \deg(\Gamma) \cdot m - \deg(I_p) - \delta(\Gamma) \end{aligned}$$

using that by (2) the ideal I_p is an adjoint ideal of Γ_p . By (4) the chain of inequalities is an equality, hence

$$\tilde{d}(g_p) = d(g_p) = \deg(\Gamma_p) \cdot m - \deg \Delta(I_p)$$

and

$$\delta(\Gamma) = \delta(\Gamma_p).$$

By (8.2) and Lemma 3.13 this implies that

$$\deg \Delta(I_p) = \deg(I_p) + \delta(\Gamma_p) = \deg(I) + \delta(\Gamma) \geq \deg \Delta(I), \quad (8.3)$$

or equivalently

$$d(g_p) \leq d(g).$$

To prove equality, we consider the closed subscheme

$$\mathcal{X} = V(I(\Gamma)_0) \subset \mathbb{P}_{\mathbb{Z}}^r \xrightarrow{\pi} \text{Spec } \mathbb{Z}$$

with projection π and fibers $\mathcal{X}_q = \pi^{-1}(\langle q \rangle) = \mathcal{X} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \kappa(\langle q \rangle)$. So over the generic point $\langle 0 \rangle \in \text{Spec } \mathbb{Z}$ the fiber is $\mathcal{X}_0 = \Gamma$ and over $\langle p \rangle$ it is $\mathcal{X}_p = \Gamma_p$. By (1) the Hilbert polynomials of Γ and Γ_p are equal, hence there is a Zariski open subset $V \subset \text{Spec } \mathbb{Z}$ with $\langle p \rangle \in V$ such that the Hilbert polynomial is constant on V . So $\pi_V : \mathcal{X}_V = \pi^{-1}(V) \rightarrow V$ is a flat family (see [Hartshorne 1977, Ch. III, Thm. 9.9]).

Since $\delta(\Gamma_p) = \delta(\Gamma)$, the δ -constant criterion for simultaneous normalization (see [Lipman 2006]) implies that there is a Zariski open subset $U \subset V \subset \text{Spec } \mathbb{Z}$ with $\langle p \rangle \in U$ such that π_U is equinormalizable. That is, there is a finite map $\nu : Z \rightarrow \mathcal{X}_U$ such that $\bar{\pi} := \pi_U \circ \nu$ is flat with nonempty geometrically normal fibers, and for each $\langle q \rangle \in U$ the induced map on the fibers $\nu_q : \bar{\mathcal{X}}_q = \bar{\pi}^{-1}(\langle q \rangle) \rightarrow \pi^{-1}(\langle q \rangle) = \mathcal{X}_q$ is a normalization map.

Since, by construction, the family of sheaves defined by I_0 is flat over U and U contains both $\langle 0 \rangle$ and $\langle p \rangle$, the semicontinuity theorem (see, for example, [Liu 2002, Ch. 5, Thm. 3.20]) implies that the dimensions of the linear series induced by I on $\bar{\Gamma}$ and by I_p on $\bar{\Gamma}_p$ satisfy

$$h^0(\bar{\Gamma}_p, \mathcal{O}_{\bar{\Gamma}_p}(m \cdot H_p - \Delta(I_p))) \geq h^0(\bar{\Gamma}, \mathcal{O}_{\bar{\Gamma}}(m \cdot H - \Delta(I))).$$

Hence by (5), Riemann-Roch, and $\delta(\Gamma_p) = \delta(\Gamma)$ it follows that the degrees of the linear series satisfy $d(g_p) \geq d(g)$, so we obtain the second equality in

$$\tilde{d}(g_p) = d(g_p) = d(g)$$

(having shown the first already above). The second equality also translates into $\deg \Delta(I_p) = \deg \Delta(I)$ which, by (8.3), implies that I is an adjoint ideal. Moreover,

$$\deg(\Gamma) \cdot m - \deg \Delta(I) = \deg(\Gamma_p) \cdot m - \deg \Delta(I_p) = \tilde{d}(g_p).$$

□

Remark 8.16. Suppose now, in addition to the previous assumptions, that I_p is the Gorenstein adjoint ideal of Γ_p . Since I is an adjoint ideal of Γ , we have $I \subset \mathfrak{G}$ which implies $\deg I \geq \deg \mathfrak{G}$, hence

$$\deg \Delta(\mathfrak{G}) = \deg(\mathfrak{G}) + \delta(\Gamma) \leq \deg(I) + \delta(\Gamma) = \deg \Delta(I) = \deg \Delta(I_p). \quad (8.4)$$

Moreover, by semicontinuity

$$\dim |m \cdot H_p - \Delta(I_p)| \geq \dim |m \cdot H - \Delta(\mathfrak{G})|$$

for m large enough, so by Riemann-Roch and $\delta(\Gamma_p) = \delta(\Gamma)$ we have

$$\deg \Delta(I_p) \leq \deg \Delta(\mathfrak{G}).$$

Hence (8.4) is an equality and implies

$$\deg I = \deg \mathfrak{G},$$

that is, I is the Gorenstein adjoint ideal of Γ .

In order to expect condition (4) to be satisfied for randomly chosen g and p , the degree m has to be chosen large enough such that $\tilde{d}(g) = \deg(\Gamma) \cdot m - \deg \Delta(\mathfrak{G})$ for a generic $g \in \mathfrak{G}_m$ (taking into account that $\tilde{d}(g) = \tilde{d}(g_p)$, and $\delta(\Gamma) = \delta(\Gamma_p)$ and $I_p = \mathfrak{G}_p$, hence $\deg \Delta(I_p) = \deg \Delta(\mathfrak{G})$ holds true for all but finitely many primes p). The following lemma specifies an appropriate bound for m , which will also be sufficient to obtain (5).

Lemma 8.17. *Consider an integer m such that $P_\Gamma(m) - 1 \geq p_a(\Gamma)$ and suppose that $g \in \mathfrak{G}_m$ is generic. Then*

$$\tilde{d}(g) = \deg(\Gamma) \cdot m - \deg \Delta(\mathfrak{G})$$

Furthermore, $|mH - \Delta(\mathfrak{G})|$ is nonspecial.

Proof. By assumption and since $P_\Gamma(m) = (\deg \Gamma) \cdot m - p_a(\Gamma) + 1$, we have

$$\deg(\Gamma) \cdot m \geq 2p_a(\Gamma).$$

By Corollary 3.19, we obtain $\deg \Delta(\mathfrak{G}) \leq 2\delta(\Gamma)$. Hence, it follows that

$$\begin{aligned} \deg(\Gamma) \cdot m - \deg \Delta(\mathfrak{G}) &\geq \deg(\Gamma) \cdot m - 2\delta(\Gamma) \\ &= \deg(\Gamma) \cdot m - 2p_a(\Gamma) + 2p(\Gamma) \geq 2p(\Gamma). \end{aligned}$$

This implies that $|mH - \Delta(\mathfrak{G})|$ is base-point free (see [Hartshorne 1977, Ch. IV, Cor. 3.2]), hence, since g is generic, we have $d(g) = \tilde{d}(g)$. By reason of its degree, the linear series is also nonspecial (see [Hartshorne 1977, Ch. IV, Ex. 1.3.4]). □

Remark 8.18. For a plane curve Γ of degree n the condition $P_\Gamma(m) - 1 \geq p_a(\Gamma)$ is equivalent to $n \cdot m \geq (n-1)(n-2)$, which is satisfied for $m \geq n-2$.

We summarize our approach in Algorithm 6.

Algorithm 6 Modular adjoint ideal

Input: A curve $\Gamma \subset \mathbb{P}^r$ satisfying the conditions of Notation 8.1.

Output: The Gorenstein adjoint ideal $\mathfrak{G}(\Gamma)$.

```

1: choose an integer  $t \geq 1$ 
2:  $\mathcal{P} = \mathcal{GP} = \emptyset$ 
3: loop
4:   choose a list  $\mathcal{Q}$  of  $t$  random primes not used so far
5:   for all  $p \in \mathcal{Q}$  do
6:     if  $\Gamma_p$  is irreducible, non-degenerate, and  $\text{LM}(I(\Gamma)) = \text{LM}(I(\Gamma_p))$  then
7:       compute the reduced Gröbner basis  $G_p$  of  $\mathfrak{G}(\Gamma_p) \subset R_p$  (via Alg. 3)
8:        $\mathcal{P} = \mathcal{P} \cup \{p\}$ ,  $\mathcal{GP} = \mathcal{GP} \cup \{G_p\}$ 
9:        $(\mathcal{GP}, \mathcal{P}) = \text{DELETEBYMAJORITYVOTE}(\mathcal{GP}, \mathcal{P})$ 
10:    lift  $(\mathcal{GP}, \mathcal{P})$  to a set of polynomials  $G \subset R$  via the Chinese remainder theorem
        and Gaussian reduction
11:    if the lifting succeeds and  $\text{PTEST}(I(\Gamma), G, \mathcal{P})$  then
12:      if  $G$  is a Gröbner basis and  $\langle G \rangle$  is saturated and  $\langle G \rangle : M^\infty = \langle 1 \rangle$  then
13:        choose  $m$  such that  $P_\Gamma(m) - 1 \geq p_a(\Gamma)$ 
14:        choose  $g \in \langle G \rangle_m$  at random
15:        choose a prime  $p \in \mathcal{P}$ 
16:        compute  $\tilde{d}(g_p) = \deg((I(\Gamma_p) + \langle g_p \rangle) : M_p^\infty)$ 
17:        compute  $\delta(\Gamma)$  by applying Remark 7.3
18:        if  $\tilde{d}(g_p) = \deg(\Gamma) \cdot m - \deg \langle G(p) \rangle - \delta(\Gamma)$  then
19:          return  $\langle G \rangle$ 

```

Remark 8.19. In Algorithm 6, the different $G(p)$ can be computed in parallel. The individual computations can be parallelized by partitioning the singular loci.

Remark 8.20. The most expensive step of the verification is the computation of $\delta(\Gamma)$. If we skip the verification, the algorithm will become probabilistic, that is, the output is the Gorenstein adjoint ideal only with high probability. This usually accelerates the algorithm considerably and gives us, in particular, a fast probabilistic way to compute both the geometric genus $p(\Gamma)$ and $\deg \Delta(\mathfrak{G}) = \dim_{\mathbb{Q}} \left(\overline{\mathbb{Q}[C]} / \mathcal{C}_{\mathbb{Q}[C]} \right)$.

9. TIMINGS

The algorithms for adjoint ideals presented in this paper are implemented in the SINGULAR library `adjointideal.lib` (see [Böhm et al. 2015b]). They make use of the normalization algorithm of Section 2 either in its local or local to global variant, as appropriate. These variants, in turn, are part of the SINGULAR library `locnormal.lib` (see [Böhm et al. 2012b]).

In this section, we compare the performance of the different algorithms. Specifically, we consider

LA	Mnuk's global linear algebra approach (Algorithm 1),
IQ	the global ideal quotient approach (Algorithm 2),
locIQ	the local ideal quotient approach (Algorithm 3 using Algorithm 4),
locIQP2	the local ideal quotient approach for plane curves with the improvements of Section 7 concerning ordinary multiple points and singularities of type <i>ADE</i> , and
modLocIQ	the modular local ideal quotient strategy (Algorithm 6).

For the modular approach, we do not make use of a local analysis of the singular locus except for computing the invariants needed in the verification step.

To quantify the improvement in computation time obtained by omitting the verification step in the modular approach, we give timings for the *resulting, now probabilistic, version of Algorithm 6* (denoted by `modLocIQ'` in the tables). In all examples computed so far, the result of the probabilistic algorithm is indeed correct.

To quantify the contributions of the different normalization algorithms and to provide a lower bound for any adjoint ideal algorithm using them, we also specify the following computation times: normalization in SINGULAR via the local to global approach outlined in Section 2 (denoted by `locNormal`); and finding an integral basis in MAPLE via the algorithm of van Hoeij (denoted by `Maple-IB`). Once being fully implemented in SINGULAR, we expect further improvements of the performance by computing the local contribution or just an integral basis of the local ring by the algorithm discussed in [Böhm et al. 2015a]. Since this algorithm and van Hoeij's algorithm rely on Puiseux series, they work in characteristic zero only.

All timings are in seconds on an AMD Opteron 6174 machine with 48 cores, 2.2GHz, and 128GB of RAM running a Linux operating system. A dash indicates that the computation did not finish within 10000 seconds. The *timings for parallel computations are marked by the symbol ** and the *maximum number of cores used in parallel is indicated in brackets*.

Remark 9.1. All examples are defined over the field of rationals. For `locIQ*`, the number of cores used corresponds to the number of components of the decomposition of the singular locus over \mathbb{Q} . For `modLocIQ*`, the number of cores used in a given iteration of the algorithm is obtained by summing up the number of components modulo p over all primes $p \in \mathcal{Q}$ chosen in Step 4 of Algorithm 6.

To show the power of the modular algorithm, we give simulated parallel timings even if the number of processes exceeds the number of cores available on our machine (which is a valid approach since the algorithm has basically zero communication overhead). For the single-core timings of `modLocIQ`, we *indicate in square brackets the number of primes used by the algorithm*.

Now we turn to explicit examples. First we consider rational plane curves defined by a random parametrization of degree n . These curves have $\binom{n-1}{2}$ ordinary double points. Their defining equations $f_{1,n}$ were generated by the function `randomRatCurve` from the SINGULAR library `paraplanecurves.lib` (see [Böhm et al. 2012c]), using the random seed 1 and a random parametrization with coefficients of bitlength 15.

	$f_{1,5}$	$f_{1,6}$	$f_{1,7}$
deg	5	6	7
<code>locNormal</code>	2.1	56	—
<code>Maple-IB</code>	5.1	47	318
LA	98	4400	—
IQ	2.1	56	—
<code>locIQ</code>	1.3	54	3800
<code>locIQ*</code>	1.3 (1)	54 (1)	3800 (1)
<code>locIQP2</code>	.18	1.2	49
<code>locIQP2*</code>	.18 (1)	1.2 (1)	49 (1)
<code>modLocIQ</code>	6.4 [33]	19 [53]	150 [75]
<code>modLocIQ'</code>	6.2 [33]	18 [53]	104 [75]
<code>modLocIQ*</code>	.36 (74)	1.6 (153)	51 (230)
<code>modLocIQ'*</code>	.21 (74)	0.48 (153)	5.2 (230)

We observe that the detection of special types of singularities is fast and yields the best performance among the non-probabilistic algorithms.

To compare the algorithms at a single singularity, we consider plane curves with exactly one A_n respectively D_n singularity at the origin of the affine chart $\{Z \neq 0\}$ (ignoring singularities at infinity). For the modular approach, we omit verification since this step relies on global properties of the curve.

The curves with affine equation $f_{2,n,d} = Y^2 + X^{n+1} + Y^d$, $n \geq 1$, $d \geq 3$, have precisely one singularity of type A_n at the origin:

	$f_{2,5,10}$	$f_{2,5,100}$	$f_{2,5,500}$	$f_{2,50,100}$	$f_{2,50,500}$	$f_{2,400,500}$
deg	10	100	500	100	500	500
locNormal	.12	.12	.12	.51	.51	3.6
Maple-IB	.08	1.5	96	4.7	150	630
LA	.18	140	—	150	—	—
IQ	.12	.12	.12	.51	.51	3.6
modLocIQ'	.20 [2]	.22 [2]	.96 [2]	1.1 [2]	2.0 [2]	11 [2]
modLocIQ'*	.10 (2)	.13 (2)	.48 (2)	.54 (2)	1.2 (2)	5.8 (2)

The curves with affine equation $f_{3,n,d} = X(X^{n-1} + Y^2) + Y^d$, $n \geq 2$, $d \geq 3$, have exactly one singularity of type D_n at the origin:

	$f_{3,5,10}$	$f_{3,5,100}$	$f_{3,5,500}$	$f_{3,50,100}$	$f_{3,50,500}$	$f_{3,400,500}$
deg	10	100	500	100	50	500
locNormal	.15	.15	.15	.67	.67	4.9
Maple-IB	.05	1.7	100	34	1830	—
LA	.20	140	—	140	—	—
IQ	.15	.15	.15	.67	.67	5.0
modLocIQ'	.22 [2]	.23 [2]	.23 [2]	1.5 [2]	1.5 [2]	24 [2]
modLocIQ'*	.09 (2)	.10 (2)	.10 (2)	.74 (2)	.77 (2)	17 (2)

In both examples, the best strategy is IQ since we consider only one singularity and since no coefficients of large bitlength occur.

The plane curves with defining equations

$$f_{4,n} = (X^{n+1} + Y^{n+1} + Z^{n+1})^2 - 4(X^{n+1}Y^{n+1} + Y^{n+1}Z^{n+1} + Z^{n+1}X^{n+1})$$

were given in [Hirano 1992] and have $3(n+1)$ singularities of type A_n if n is even. To ensure that all singularities of the curves are in the affine chart $\{Z \neq 0\}$, we substitute $Z = 2X - 3Y + 1$.

	$f_{4,4}$	$f_{4,6}$	$f_{4,8}$
deg	10	14	18
locNormal	1.6	—	—
Maple-IB	2.2	14	70
LA	89	—	—
IQ	2.5	—	—
locIQ	.96	—	—
locIQ*	.36 (6)	—	—
locIQP2	1.0	—	—
locIQP2*	.38 (6)	—	—
modLocIQ	3.7 [3]	23 [4]	190 [4]
modLocIQ'	3.3 [3]	20 [4]	170 [4]
modLocIQ*	.63 (27)	4.4 (48)	50 (48)
modLocIQ'*	.38 (27)	2.2 (48)	30 (48)

To conclude this section, we present examples of curves in higher-dimensional projective space. As above, we first consider curves with only one singularity in a given affine chart: let L_n be the ideal of the image of

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^3, t \mapsto (t^{n-2}, t^{n-1}, t^n).$$

Second, denote by I_n the ideal of the image in \mathbb{P}^5 under the degree-2 Veronese embedding of the curve $\{f_{4,n} = 0\}$. The resulting timings are:

	L_{25}	L_{50}	I_4	I_6
deg	25	50	20	28
locNormal	3.9	84	21	—
IQ	3.9	84	30	—
locIQ	3.9	84	18	—
locIQ*	3.9 (1)	84 (1)	7.5 (6)	—
modLocIQ'	6.5 [2]	220 [2]	74 [5]	2600 [5]
modLocIQ',*	3.3 (2)	140 (2)	4.0 (45)	59 (69)

To summarize, we observe that the ideal quotient approach is faster than the linear algebra one. To some extent, this is due to the lack of efficiency of the rational function arithmetic in SINGULAR. The local strategy is faster than the global one if there is more than one component in the decomposition of the singular locus over \mathbb{Q} . In addition, the local algorithm can be run in parallel and is, then, even faster. In most examples, especially when the coefficients have large bitlength, the fastest approach is the modular local strategy, which parallelizes in a two-fold way, by localization and modularization. In contrast to other modular algorithms (such as modular normalization), the verification step is usually very fast.

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